Maximum Principle for Reflected BSPDE and Mean Field Game Theory with Applications

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von
Guanxing Fu M.Sc.

Präsidentin der Humboldt-Universität zu Berlin
Prof. Dr.-Ing. Dr. Sabine Kunst

Dekan der Mathematisch-Naturwissenschaftlichen Fakultät
Prof. Dr. Elmar Kulke

Gutachter:
1. Prof. Dr. Ulrich Horst (Humboldt-Universität zu Berlin)
2. Prof. Dr. Jean-Pierre Fouque (University of California, Santa Barbara)
3. Prof. Dr. Huyên Pham (University Paris Diderot (Paris 7))

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To my family

Abstract

The thesis is concerned with two topics: backward stochastic partial differential equations and mean filed games.

In the first part, we establish a maximum principle for quasi-linear reflected backward stochastic partial differential equations (RBSPDEs) on a general domain by using a stochastic version of De Giorgi's iteration. The maximum principle for RBSPDEs on a bounded domain and the maximum principle for BSPDEs on a general domain are obtained as byproducts. Finally, the local behavior of the weak solutions is considered.

In the second part, we first establish the existence of equilibria to mean field games (MFGs) with singular controls. We also prove that the solutions to MFGs with no terminal cost and no cost from singular controls can be approximated by the solutions, respectively control rules, for MFGs with purely regular controls. Our existence and approximation results strongly hinge on the use of the Skorokhod M_1 topology on the space of càdlàg functions.

Subsequently, we consider an MFG of optimal portfolio liquidation under asymmetric information. We prove that the solution to the MFG can be characterized in terms of a forward backward stochastic differential equation (FBSDE) with possibly singular terminal condition on the backward component or, equivalently, in terms of an FBSDE with finite terminal value, yet singular driver. We apply the fixed point argument to prove the existence and uniqueness on a short time horizon in a weighted space. Our existence and uniqueness result allows to prove that our MFG can be approximated by a sequence of MFGs without state constraint.

The final result of the second part is a leader follower MFG with terminal constraint arising from optimal portfolio liquidation between hierarchical agents. We show the problems for both follower and leader reduce to the solvability of singular FBSDEs, which can be solved by a modified approach of the previous result.

Zusammenfassung

Diese Arbeit behandelt zwei Gebiete: stochastische partielle Rückwerts-Differentialgleichungen (BSPDEs) und Mean-Field-Games (MFGs).

Im ersten Teil wird über eine stochastische Variante der De Giorgischen Iteration ein Maximumprinzip für quasilineare reflektierte BSPDEs (RBSPDEs) auf allgemeinen Gebieten bewiesen. Als Folgerung erhalten wir ein Maximumprinzip für RBSPDEs auf beschränkten, sowie für BSPDEs auf allgemeinen Gebieten. Abschließend wird das lokale Verhalten schwacher Lösungen untersucht.

Im zweiten Teil zeigen wir zunächst die Existenz von Gleichgewichten in MFGs mit singulärer Kontrolle. Wir beweisen, dass die Lösung eines MFG ohne Endkosten und ohne Kosten in der singulären Kontrolle durch die Lösungen eines MFGs mit strikt regulären Kontrollen approximiert werden kann. Die vorgelegten Existenz- und Approximationsresultat basieren entscheidend auf der Wahl der Storokhod M_1 Topologie auf dem Raum der Càdlàg-Funktion.

Anschließend betrachten wir ein MFG optimaler Portfolioliquidierung unter asymmetrischer Information. Die Lösung des MFG charakterisieren wir über eine stochastische Vorwärts-Rückwärts-Differentialgleichung (FBSDE) mit singulärer Endbedingung der Rückwärtsgleichung oder alternativ über eine FBSDE mit endlicher Endbedingung, jedoch singulärem Treiber. Wir geben ein Fixpunktargument, um die Existenz und Eindeutigkeit einer Kurzzeitlösung in einem gewichteten Funktionenraum zu zeigen. Dies ermöglicht es, das ursprüngliche MFG mit entsprechenden MFGs ohne Zustandsendbedinung zu approximieren.

Der zweite Teil wird abgeschlossen mit einem Leader-Follower-MFG mit Zustandsendbedingung im Kontext optimaler Portfolioliquidierung bei hierarchischer Agentenstruktur. Wir zeigen, dass das Problem beider Spielertypen auf singuläre FBSDEs zurückgeführt werden kann, welche mit ähnlichen Methoden wie im vorangegangen Abschnitt behandelt werden können.

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Contents

1. Introduction

There are three main approaches to solve stochastic optimal control problems: the Pontryagin maximum principle, the dynamic programming principle and the compactification method. The main idea behind the first method is, by calculus of variations, to reduce the solvability of the control problem to the solvability of a forward-backward stochastic differential equation; the main idea behind the second method is, by backward induction, to reduce the control problem to the solvability of a backward equation (partial differential equation, backward stochastic differential equation or backward stochastic partial differential equation). The compactification method is also called the relaxed solution method. The method is based on the idea that an upper (lower) semi-continuous function attains its maximum (minimum) on a compact set. The aim is thus to prove that the original (unconstrained) optimization problem is equivalent to a constrained problem where the set of admissible controls is constrained to a compact set, and the cost functional is semi-continuous.

This thesis contributes to the theory of backward stochastic partial differential equations (BSPDEs) and forward-backward stochastic differential equations (FBSDEs) arising in problems of optimal control and to the theory of mean field games (MFGs). In Chapter 2, we establish a maximum principle for a broad class of nonlinear reflected BSPDEs (RBSPDEs). In Chapter 3, by an adaption of the compactification method, we establish novel existence of equilibrium results for MFGs with singular controls. In Chapter 4 and Chapter 5 we analyze a class of MFGs and McKean-Vlasov FBSDEs with state constraints that arise in models of optimal portfolio liquidation under strategic interaction.

1.1. Part I: Maximum Principle for RBSPDE

Since their introduction by Bensoussan [Ben83], BSPDEs have been extensively investigated in the probability and stochastic control literature. They naturally arise in many applications, for instance as stochastic Hamilton-Jacobi-Bellman equations associated with non-Markovian control problems [Pen92], as adjoint equations of the Duncan-Mortensen-Zakai equation in nonlinear filtering [Zho92] and as adjoint equations in stochastic control problems when formulating stochastic maximum principles [Ben83]. BSPDEs with singular terminal conditions arise in non-Markovian models for financial mathematics to describe optimal trading in illiquid financial markets [GHQ15].

RBSPDEs arise as the Hamilton-Jacobi-Bellman equation for the optimal stopping problem of stochastic differential equations with random coefficients [CPY09, QW14], and as the adjoint equations for the maximum principle of Pontryagin type in singular control problems of stochastic partial differential equations (SPDEs) in, e.g. [ØSZ13].

Existence and uniqueness of solutions results for reflected PDEs and SPDEs have been established by many authors. Pierre [Pie79, Pie80] has studied parabolic PDEs with obstacles using parabolic potentials. Using methods and techniques from parabolic potential theory Denis, Matoussi and Zhang [DMZ14b] proved existence and uniqueness of solutions results for quasi-linear SPDEs driven by infinite dimensional Brownian motion. More recently, Qiu and Wei [QW14] established a general theory of existence and uniqueness of solutions for a class of quasi-linear RBSPDEs, which includes the classical results on obstacle problems for deterministic parabolic PDEs as special cases.

Adapting Moser's iteration scheme to the nonlinear case, Aronson and Serrin [AS67] proved the maximum principle and local bounds of weak solutions for deterministic quasi-linear parabolic equations on bounded domains. Their method was extended by Denis, Matoussi, and Stoica [DMS05] to the stochastic case, obtaining an L^p a priori estimate for the uniform norm to solutions of the stochastic quasi-linear parabolic equation with null Dirichlet condition. It was further adapted by Denis, Matoussi, and Stoica [DMS09] to local solutions. Later, Denis, Matoussi, and Zhang [DMZ14a] established L^p estimates for the uniform norm in time and space of weak solutions to reflected quasi-linear SPDEs along with a maximum principle for local solutions using a stochastic version of Moser's iteration scheme. Recently, Qiu and Tang [QT12] used the De Giorgi's iteration scheme, a technique that also works for degenerate parabolic equations, to establish a local and global maximum principle for weak solutions of BSPDEs without reflection.

In Chapter 2 we establish a maximum principle for RBSPDEs on possibly unbounded domains. A maximum principle and a comparison principle for BSPDEs on general domains, a maximum principle for RBSPDEs on bounded domains and a local maximum principle for RBSPDEs are obtained as well. Due to the obstacle, the maximum principle for RBSPDEs is not a direct extension of that for BSPDEs in [QT12]. Our proofs rely on a stochastic version of De Giorgi's iteration scheme that does not depend on the Lebesgue measure of the domain; this extends the scheme in [QT12] that only applies to bounded domains. Our iteration scheme requires an almost sure representation of the L^2 norm of the positive part of the weak solution to RBSPDEs, which is obtained through a generalization of the Itô's formula for weak solutions to BSPDEs.

Chapter 2 is based on the paper [FHQ17]. In that paper we consider the following quasi-linear RBSPDE:

$$\begin{cases} -du(t,x) = [\partial_j(a^{ij}\partial_i u(t,x) + \sigma^{jr}v^r(t,x)) + f(t,x,u(t,x),\nabla u(t,x),v(t,x)) \\ \qquad\qquad + \nabla \cdot g(t,x,u(t,x),\nabla u(t,x),v(t,x))]\,dt + \mu(dt,x) - v^r(t,x)dW_t^r, \\ \qquad\qquad (t,x) \in Q := [0,T]\times\mathcal{O}, \\ u(T,x) = G(x), \quad x \in \mathcal{O}, \\ u(t,x) \geq \xi(t,x)\ dt\times dx\times d\mathbb{P} - a.e., \\ \int_Q (u(t,x)-\xi(t,x))\mu(dt,dx) = 0, \end{cases} \tag{1.1}$$

where $\mathcal{O}$ is a general domain in $\mathbb{R}^n$. The following maximum principles are the main results of Chapter 2 (see Theorem 2.3.1 and Theorem 2.3.10, respectively).

Main Result 1. If the triplet (u,v,μ) is a solution to the RBSPDE (1.1), then

$$\begin{aligned} &\mathrm{esssup}_{(\omega,t,x)\in\Omega\times Q} u^{\pm} \\ \leq & C\left(\mathrm{esssup}_{(\omega,t,x)\in\Omega\times\partial_p Q} u^{\pm} + \mathrm{esssup}_{(\omega,t,x)\in\Omega\times\partial_p Q}\hat{\xi}^{\pm} + \mathcal{C}(f,g,\xi)\right), \end{aligned}$$

where $\partial_p Q$ is the parabolic boundary of Q, the positive constant C depends only on the bound related to the coefficients, dimension and the time horizon of the equation, and $\mathcal{C}(f,g,\xi)$ is a constant depending on the coefficients f,g and the obstacle ξ.

Main Result 2. Let (u,v,μ) be a solution to the RBSPDE (1.1). For any fixed $(t_0,x_0)\in[0,T]\times\mathbb{R}^n$, given $Q_{2\rho} := [t_0-4\rho^2,t_0)\times B_{2\rho}(x_0)\subset Q$ with $\rho\in(0,1)$, we have

$$\mathrm{esssup}_{(\omega,s,x)\in\Omega\times Q_{\frac{\rho}{2}}} u^{\pm} \leq C\left\{\rho^{-\frac{n+2}{2}}(\|u^{\pm}\|_{0,2;Q_\rho} + \|\hat{\xi}^{\pm}\|_{0,2;Q_{2\rho}}) + \widetilde{\mathcal{C}}(f,g,\xi,\rho)\rho^{1-\frac{2+n}{p}}\right\},$$

where C is a positive constant depending on the bound related to the coefficients, dimension and time horizon of the problem, and $\widetilde{\mathcal{C}}(f,g,\xi,\rho)$ is a constant depending on the coefficients f,g, the obstacle ξ and ρ.

It is worth pointing out that by contrast to L^p estimates ($p\in(2,\infty)$) for the time and space maximal norm of weak solutions to *forward* SPDEs or related obstacle problems as established in [DMS05, DMZ14a, Qiu15], our estimate for weak solutions is uniform with respect to $w\in\Omega$ and hence establishes an L^∞ estimate. This distinction comes from the essential difference between BSPDEs and *forward* SPDEs: the noise term in the former endogenously originates from martingale representation and is hence governed by the coefficients, while the noise term in the latter is fully exogenous, which prevents any L^∞ estimate for *forward* SPDEs.

1.2. Part II: Mean Field Game Theory and Its Application to Optimal Portfolio Liquidation

The second part of this thesis (Chapter 3, Chapter 4 and Chapter 5) is concerned with MFG theory and its application to portfolio liquidation.

MFGs are a powerful tool to analyse strategic interactions in large populations when each individual player has only a small impact on the behavior of other players. In the economics literature, MFGs (or anonymous games) were first considered by Jovanovic and Rosenthal [JR88]. Anonymous and mean field type games were subsequently analyzed in the economics literature by many authors including Blonski [Blo99, Blo00], Daskalakis and Papadimitriou [DP15], Horst [Hor05], and Rath [Rat96]. Applications of MFGs in mathematical economics and finance range from models of optimal exploitation of exhaustible resources [CS15, CS17, GLL11] to systemic risk [CFS15, CFMS16], from bank run models [CDL17, Nut17] to portfolio optimization [LZ17], and from principal-agent problems [EMP16] to problems of optimal trading under market impact [CL15, HJN15, CL17]. In the mathematical literature they were independently introduced by Huang, Malhamé and Caines [HMP06] as well as Lasry and Lions [LL07].

In a standard MFG as considered in [HMP06, LL07], each player $i \in \{1, ..., N\}$ chooses an action from a given set of admissible controls that minimizes a cost functional of the form

$$J^i(u) = \mathbb{E}\left[\int_0^T f(t, X_t^i, \bar{\mu}_t^N, u_t^i)dt + g(X_T^i, \bar{\mu}_T^N)\right] \tag{1.2}$$

subject to the state dynamics

$$\begin{cases} dX_t^i = b(t, X_t^i, \bar{\mu}_t^N, u_t^i)\, dt + \sigma(t, X_t^i, \bar{\mu}_t^N, u_t^i)\, dW_t^i, \\ X_0^i = x_0 \end{cases} . \tag{1.3}$$

Here $W^1, \cdots, W^N$ are independent Brownian motions defined on some underlying filtered probability space, $X^i \in \mathbb{R}^d$ is the *state* of player i, $u = (u^1, \cdots, u^N)$, $u^i = (u_t^i)_{t\in[0,T]}$ is an adapted stochastic process, the *action* of player i, and $\bar{\mu}_t^N := \frac{1}{N}\sum_{j=1}^N \delta_{X_t^j}$ denotes the empirical distribution of the individual players' states at time $t \in [0, T]$.

The existence of *approximate Nash equilibria* in the above game for large populations has been established in [CD13, HMP06] using a representative agent approach. In view of the independence of the Brownian motions the idea to solve the problem is to first approximate the dynamics of the empirical distribution by a deterministic measure-valued process and to consider instead the optimization problem of a representative player that takes the distribution of the states as given, and then to solve the fixed-point problem of finding a measure-valued process such that the distribution of the representative player's state process X under her optimal strategy coincides with that process. The idea of decoupling local from global dynamic in large population has been applied to equilibrium models of social interaction in e.g. [HS06, HS09].

Following the representative agent approach, an MFG can then be formally described by a coupled optimization and fixed point problem of the form:

$$\begin{cases} 1. \quad \text{fix a deterministic function } t \in [0,T] \mapsto \mu_t \in \mathcal{P}(\mathbb{R}^d); \\ 2. \quad \text{solve the corresponding stochastic control problem :} \\ \qquad \inf_u \mathbb{E}\left[\int_0^T f(t, X_t, \mu_t, u_t)\, dt + g(X_T, \mu_T)\right], \\ \qquad \text{subject to} \\ \qquad dX_t = b(t, X_t, \mu_t, u_t)\, dt + \sigma(t, X_t, \mu_t, u_t)\, dW_t \\ \qquad X_0 = x_0, \\ 3. \quad \text{solve the fixed point problem } Law(X) = \mu \\ \qquad \text{where } X \text{ is the optimal state process from 2.} \end{cases} \tag{1.4}$$

Here, $\mathcal{P}(\mathbb{R}^d)$ is the space of probability measures on $\mathbb{R}^d$ and $Law(X)$ denotes the law of the process X.

There are essentially three approaches to solve MFGs. In their original paper [LL07], Lasry and Lions followed an analytic approach. They analyzed a coupled forward-backward PDE system, where the backward component is the Hamiltion-Jacobi-Bellman equation arising from the representative agent's optimization problem, and the forward component is a Kolmogorov-Fokker-Planck equation that characterizes the dynamics of the state process.

A second, more probabilistic, approach was introduced by Carmona and Delarue in [CD13]. Using a maximum principle of Pontryagin type, they showed that the fixed point problem reduces to solving a McKean-Vlasov FBSDE. [BSYY16, CDL13] consider linear-quadratic MFGs, while [Ahu16, CZ16] consider MFGs with common noise and with major and minor players, respectively. A class of MFGs in which the interaction takes place through both the state dynamics and the controls has recently been introduced in [CL15]. In that paper the martingale optimality principle is used to prove the existence of a solution.

A relaxed solution concept to MFGs was introduced by Lacker in [Lac15]. Considering MFGs from a more game-theoretic perspective, the idea is to search for equilibria in relaxed controls ("mixed strategies") by first establishing the upper hemi-continuity of the representative agent's best response correspondence to a given μ using Berge's maximum theorem, and then to apply the Kakutani-Fan-Glicksberg fixed point theorem in order to establish the existence of some measure-valued process μ^* such that the law of the agent's state process under a best response to μ^* coincides with that process. Relaxed controls date back to Young [You37]. They were later applied to stochastic control in e.g. [HL90, HS95, EKDHJP87], to MFGs in [Lac15], and to MFGs with common noise in [CDL16].

1.2.1. Summary of Chapter 3

Chapter 3 is based on the paper [FH17]. In that paper we establish the existence of relaxed solutions to MFGs with singular controls of the form

$$\begin{cases} 1. & \text{fix a deterministic function } t \in [0,T] \mapsto \mu_t \in \mathcal{P}(\mathbb{R}^d); \\ 2. & \text{solve the corresponding stochastic singular control problem :} \\ & \inf_{u,Z} \mathbb{E}\left[\int_0^T f(t,X_t,\mu_t,u_t)\,dt + g(X_T,\mu_T) + \int_0^T h(t)\,dZ_t\right], \\ & \text{subject to} \\ & dX_t = b(t,X_t,\mu_t,u_t)\,dt + \sigma(t,X_t,\mu_t,u_t)\,dW_t + c(t)\,dZ_t, \\ 3. & \text{solve } Law(X) = \mu, \text{ where } X \text{ is the optimal state process from 2.,} \end{cases}$$

where $u = (u_t)_{t\in[0,T]}$ is the *regular control* and $Z = (Z_t)_{t\in[0,t]}$ is the *singular control*. When singular controls are admissible, the state process no longer takes values in the space of continuous functions, but rather in the Skorokhod space $\mathcal{D}(0,T)$ of all càdlàg functions. The key is then to identify a suitable topology on the Skorokhod space with respect to which the compactness and continuity assumptions on the maximum and the fixed-point theorems are satisfied.

There are essentially three possible topologies on the space of càdlàg functions: the (standard) Skorokhod J_1 topology (J_1 topology for short), the Meyer-Zheng topology (or pseudo-path topology), and the Skorokhod M_1 topology (M_1 topology for short). The M_1 topology seems to be the most appropriate one for our purposes. First, the set of bounded singular controls is compact in the M_1 topology but not in the J_1 topology. Second, there is no explicit expression for the metric corresponding to Meyer-Zheng topology. In particular, one cannot bound the value of a function at given points in time by the Meyer-Zheng topology. Third, the M_1 topology has better continuity properties than the J_1 topology. For instance, it allows for an approximation of discontinuous functions by continuous ones. This enables us to approximate solutions to certain classes of MFGs with singular controls by solutions to MFGs with only regular controls. Appendix A.4 summarizes useful properties of the M_1 topology; for more details, we refer to the textbook of Whitt [Whi02].

To the best of our knowledge, ours is the first result to establish the existence of solutions results to MFGs with singular controls. The recent paper [GL17] only considers absolutely continuous singular controls. Our notion of singular controls is more general. As a byproduct, we obtain a new proof for the existence of optimal (relaxed) controls for the corresponding class of stochastic singular control problems. A similar control problem, albeit with a trivial terminal cost function, has been analyzed in [HS95]. While the methods and techniques applied therein can be extended to nontrivial terminal cost functions after a modification of the control problem, they cannot be used to prove existence of equilibria in MFGs. In fact, in [HS95], it is assumed that the state space $\mathcal{D}(0,T)$ is endowed with Meyer-Zheng topology, and that the spaces of admissible singular and regular controls are endowed with the topology of weak convergence and the stable topology, respectively. With this choice of topologies the continuity of cost functional and the upper hemi-continuity of distribution of the representative agent's state process under the optimal control w.r.t. to a given process μ cannot be established. As a second byproduct we obtain a novel existence of solutions result for a class of McKean-Vlasov stochastic singular control problems. MFGs and control problems of McKean-Vlasov type are compared in [CDL13]. The main difference between these somewhat similar, yet very different problems lies in the order of carrying out the optimization and the fixed point arguments. When optimizing first, the subsequent fixed point problem leads to MFGs, while in McKean-Vlasov control problems one searches for fixed points before solving the optimization problem. For details, refer to [CDL13].

Our second main contributions are two approximation results that allow us to approximate solutions to a certain class of MFGs with singular controls by the solutions to MFGs with only regular controls. The approximation result, too, strongly hinges on the choice of the M_1 topology.

1.2.2. Summary of Chapter 4 and Chapter 5

In the last two chapters, we analyze two MFG models of optimal portfolio liquidation. They are based on ongoing work with Paulwin Graewe, Ulrich Horst and Alexandre Popier.

Single-player portfolio liquidation models have been extensively analyzed in recent years; see [AC01, GHS17, AJK14, HN14, GHQ15, BBF16, KP16, GH17, BBF18] among others. Their main characteristic is the singularity at the terminal time of the Hamilton-Jacobi-Bellman equation. In such models the controlled state sequence typically follows a dynamics of the form

$$x_t = x - \int_0^t \xi_s \, ds,$$

where $x \in \mathbb{R}$ is the initial portfolio and ξ is the trading rate. The set of admissible controls is confined to those processes ξ that satisfy almost surely the liquidation constraint

$$x_T = 0.$$

Furthermore, it is often assumed that the unaffected benchmark price process follows a Brownian motion W (or some Brownian martingale) and that the trader's transaction price is given by

$$S_t = \sigma W_t - \int_0^t \kappa_s \xi_s \, ds - \eta_t \xi_t.$$

The integral term accounts for permanent price impact, i.e. the impact of past trades on current prices, while the term $\eta_t \xi_t$ accounts for the instantaneous impact that does not affect future transactions. The resulting expected cost functional is then of the linear-quadratic form

$$\mathbb{E}\left[\int_0^T \Big(\kappa_s \xi_s X_s + \eta_s |\xi_s|^2 + \lambda_s |x_s|^2\Big)\, ds\right],$$

where κ, η and λ are bounded adapted processes. The process λ describes the trader's degree of risk aversion; it penalizes slow liquidation. The process η describes the degree of market illiquidity; it penalizes fast liquidation. The process κ describes the impact of past trades on current transaction prices.

In Chapter 4, we analyze a novel class of MFGs arising from a game of optimal portfolio liquidation with asymmetric information between a large number N of players. Our MFGs can be characterized, equivalently, in terms of an FBSDE with a possibly singular terminal condition on the backward component, or in terms of an FBSDE with finite terminal condition yet singular driver. Specifically, the optimization problem of player $i = 1, ..., N$ is to minimize the cost functional

$$J^i(\xi) = \mathbb{E}\left[\int_0^T \left(\frac{\kappa_t^i}{N}\sum_{j=1}^N \xi_t^j X_t^i + \eta_t^i (\xi_t^i)^2 + \lambda_t^i (X_t^i)^2\right)\, dt\right] \tag{1.5}$$

subject to the state dynamics

$$dX_t^i = -\xi_t^i\, dt,\ \ X_0^i = x^i \text{ and } X_T^i = 0. \tag{1.6}$$

Here, $\xi = (\xi^1, \cdots, \xi^N)$ is the vector of strategies of each player, and κ^i, η^i and λ^i are progressively measurable with respect to the filtration

$$\mathbb{F}^i := (\mathcal{F}_t^i, 0 \le t \le T), \quad \text{with} \quad \mathcal{F}_t^i := \sigma(W_s^0, W_s^i, 0 \le s \le t).$$

We prove the existence of approximate Nash equilibria for large populations by an MFG approach. Our problem is different from standard MFGs in at least three important respects, though. First, the players interact through the impact of their strategies rather than states on the other players' payoff functions (see also [CL15]). Second, the players have private information about their instantaneous market impact, risk aversion and impact of the other players' actions on their own payoff functions. In fact, while each player's transaction price is driven by a common Brownian motion W^0, their cost coefficients are measurable functions of both the common factor W^0 and an independent idiosyncratic factor W^i. As a result, ours is an MFG with common noise (see [CDL16]). Third, and most importantly, the individual state dynamics are subject to the terminal state constraint arising from the liquidation requirement. Hence, the MFG associated with the N player game

(1.5) and (1.6) is given by:

$$\begin{cases} \text{1. fix an } \mathbb{F}^0 \text{ progressively measurable process } \mu \text{ (in some suitable space);} \\ \text{2. solve the corresponding parameterized constrained optimization problem :} \\ \inf_\xi \mathbb{E}\left[\int_0^T \left(\kappa_s \mu_s X_s + \eta_s \xi_s^2 + \lambda_s X_s^2\right)\, ds\right] \\ \ \text{s.t. } dX_t = -\xi_t\, dt,\ X_0 = x \text{ and } X_T = 0; \\ \text{3. search for the fixed point } \mu_t = \mathbb{E}[\xi_t^* | \mathcal{F}_t^0], \text{ for } a.e.\ t \in [0, T], \\ \ \text{where } \xi^* \text{ is the optimal strategy from 2.} \end{cases} \tag{1.7}$$

Here, $\mathbb{F}^0 := (\mathcal{F}_t^0, 0 \leq t \leq T)$ with $\mathcal{F}_t^0 = \sigma(W_s^0, 0 \leq s \leq t)$ and κ, η and λ are $\mathbb{F} := (\mathcal{F}_t, 0 \leq t \leq T)$ progressively measurable with $\mathcal{F}_t := \sigma(W_s^0, W_s, 0 \leq s \leq t)$.

The three papers closest to ours are Cardaliague and Lehalle [CL17], Carmona and Lacker [CL15], and Huang, Jaimungal and Nourin [HJN15]. In [CL15], the authors propose a benchmark model as a motivation to their general result. They apply a weak formulation approach to solve the problem and assume the action space to be compact. Furthermore, each player's portfolio process is subject to random fluctuations, described by independent Brownian motions. As a result, their model is much closer to a standard MFG, but no liquidation constraint is possible in their framework. The papers [CL17] and [HJN15] consider mean field models parameterized by different preferences and with major-minor players, respectively. Again, no liquidation constraint is allowed. To the best of our knowledge, ours is the first result to consider MFGs with terminal state constraint.

We apply the probabilistic method to solve the MFG with terminal constraint (1.7). In a first step we show how the analysis of our MFG can be reduced to the analysis of a conditional mean field type FBSDE. The forward component describes the optimal portfolio process; hence both its initial and terminal condition are known. The backward component describes the optimal trading rate; its terminal value is unknown and needs to be determined. Making an affine ansatz, the mean field type FBSDE with unkown terminal condition can be replaced by a coupled FBSDE with known initial and terminal condition, yet singular driver. We apply a fixed point argument to show the existence and uniqueness for problems on a short time horizon in a weighted space.

The benchmark case of constant cost coefficients can be solved in closed form. For this case we show that when the strength of interaction is large, the players initially trade very fast in equilibrium to avoid the negative drift generated by the mean field interaction. As such, our model provides a possible explanation for large price drops in markets with many strategically interacting investors.

Armed with our existence of solutions results for the MFG (1.7) we can prove that the sequence of solutions to the corresponding unconstrained penalized MFGs does indeed converge to the unique solution of the MFG (1.7) as the degree of penalization increases to infinity. The convergence result can be viewed as a consistency result for both, the unconstrained and the constrained problems. The problems should be consistent inasmuch as that the constrained problem should allow for an approximation by unconstrained problems and increasing the penalization of open positions should result in the convergence of the value functions and optimal strategies. Our approximation method also yields an alternative proof for the existence of a unique solution to the constrained problem.

By now we have considered the MFG with only one class of players. Another MFG model termed *MFG with major and minor agents* has been investigated for several years. In this model, there are two groups of players: the major one and the minor ones. The feature that all the minor players are influenced by the major one makes the problem different from the standard MFG. This model was originally proposed by [Hua10] and by [NH12] in a linear quadratic infinite horizon setting and finite horizon setting, respectively. [NC13] generalized these models to a non-linear setting. All these works consider the "mean field" as an exogenous term to the major player. For some related works, we refer to [CK17], [SC16] and [HJN15]. Later [BCY16] and [CZ16] treated the more general problem where “mean field" is endogenous to the major player with the PDE method and the probabilistic method, respectively. That means, when considering the major player’s optimization problem, the "mean field" term cannot be considered as fixed and the major player’s strategy can influence the "mean field" directly. In this case, the equilibrium between the major and minor players is of *Stackelberg type*. This is closely related to the classical leader follower stochastic differential game, see [Yon02] for a general linear quadratic case. In this game, the first participant which is called leader would send a signal to the second participant which is called the follower. By recognizing this signal, the follower chooses her strategy to optimize her cost. The leader has to know the follower’s reaction before making her decision. In analogy to the classical leader follower stochastic differential game, we prefer to call MFGs with major and minor players *leader follower MFGs*. For other results on this topic, we refer to [BCY15, CW16, BMYP17, CW17].

In Chapter 5, we consider a leader follower MFGs with constraint arising in optimal portfolio liquidation with two hierarchical groups of players. In contrast to the MFG liquidation model studied in Chapter 4, we assume that there are one major player (’leader’) and N minor players (’followers’). All the players want to liquidate their portfolio. Following the analysis of the classical leader follower game we show that the problem can be decoupled into a standard MFG and a stochastic control problem of McKean-Vlasov type.

The N player game is played among the minor players with the cost functions depending on the major player’s strategy. A generic minor player’s optimization problem is to minimize

$$\mathbb{E}\left[\int_0^T \left(\frac{\kappa_t X_t^i}{N}\sum_j^N \xi_t^j + \kappa_t^0 \xi_t^0 X_t^i + \eta_t^i(\xi_t^i)^2 + \lambda_t^i (X_t^i)^2\right) dt\right] \tag{1.8}$$

subject to

$$dX_t^i = -\xi_t^i\, dt, \qquad X_0^i = x \text{ and } X_T^i = 0.$$

The major player's cost function depends on the minor players' average action; the resulting optimization problem is to minimize

$$\mathbb{E}\left[\int_0^T \left(\frac{\kappa_t}{N}\sum_{j=1}^N \xi_t^j X_t^0 + \kappa_t^0 X_t^0 \xi_t^0 + \eta_t^0(\xi_t^0)^2 + \lambda_t^0 (X_t^0)^2 + \overline{\lambda}_t \left(\frac{1}{N}\sum_{j=1}^N \xi_t^j\right)^2\right) dt\right] \tag{1.9}$$

subject to

$$dX_t^0 = -\xi_t^0\, dt, \qquad X_0 = x^0 \text{ and } X_T^0 = 0.$$

The resulting leader-follower MFG can then be described as follows:

Step 1: *representative follower's problem.*

$$\begin{cases} \text{1. Fix a strategy of the leader } \xi^0 \text{ and a mean field } \mu; \\ \text{2. Solve the optimization problem :} \\ \quad \inf_{\xi} \mathbb{E}\left[\int_0^T \left(\kappa_t \mu_t X_t + \kappa_t^0 \xi_t^0 X_t + \eta_t \xi_t^2 + \lambda_t X_t^2\right) dt\right] \\ \quad \text{subject to } dX_t = -\xi_t\, dt, \quad X_0 = x, \quad X_T = 0; \\ \text{3. Search for the fixed point: } \mu_t = \mathbb{E}[\xi_t^* | \mathcal{F}_t^0], a.s.\ a.e., \\ \quad \text{where } \xi^* \text{ is the optimal control from 2.} \end{cases}$$

Step 2: *leader's problem.*
Let $\xi^*(\xi^0)$ and $\mu^*(\xi^0)$ be the optimal strategy and the equilibrium for the representative follower's problem, respectively. Thus, the leader's problem is to minimize

$$\mathbb{E}\left[\int_0^T \left(\kappa_t \xi_t^*(\xi^0) X_t^0 + \kappa_t^0 X_t^0 \xi_t^0 + \eta_t^0 (\xi_t^0)^2 + \lambda_t^0 (X_t^0)^2 + \overline{\lambda}_t (\mu_t^*(\xi^0))^2\right) dt\right],$$

subject to

$$\begin{cases} dX_t^0 = -\xi_t^0\, dt, \quad X_T^0 = 0, \\ \text{and the dynamics of } \xi^*(\xi^0) \text{ and } \mu^*(\xi^0). \end{cases}$$

This leads to a stochastic control problem for a McKean-Vlasov FBSDE with state constraint.

Step 1 deals with an MFG as in Chapter 4. The difference is that the MFG now depends on the leader's strategy ξ^0. Thus, the resulting FBSDE system is parameterized by ξ^0. We assume no regularity or boundedness on the trajectory of ξ^0. This renders the FBSDE different from that in Chapter 4 where the coefficients were assumed to be essentially bounded. To deal with the unboundedness of the coefficients, we introduce a different space to accommodate the solution. Moreover, in contrast to the case of essentially bounded coefficients, finer estimates are needed to cope with the conditional expectations appearing in the resulting conditional McKean-Vlasov FBSDE arising in Step 2. By making a suitable ansatz, Step

2 reduces to a 3-dimensional McKean-Vlasov FBSDE with state constraint and possibly singular terminal condition, or to a McKean-Vlasov FBSDE with known terminal conditions yet singular coefficients. We then apply similar arguments as in Chapter 4 to solve the leader-follower MFG over short time horizons.

2. PART I: Maximum Principle for Quasi-linear Reflected Backward SPDEs

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space carrying a standard m-dimensional Brownian motion $W = \{W_t, t \geq 0\}$. Let $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration generated by W, augmented by the $\mathbb{P}$-null sets in $\mathcal{F}$. In this chapter, we establish a maximum principle for weak solutions to the RBSPDE

$$\begin{cases} -du(t,x) = [\partial_j(a^{ij}\partial_i u(t,x) + \sigma^{jr}v^r(t,x)) + f(t,x,u(t,x),\nabla u(t,x),v(t,x)) \\ \qquad\qquad + \nabla\cdot g(t,x,u(t,x),\nabla u(t,x),v(t,x))]\,dt + \mu(dt,x) - v^r(t,x)dW_t^r, \\ \qquad\qquad (t,x)\in Q := [0,T]\times\mathcal{O}, \\ u(T,x) = G(x), \quad x\in\mathcal{O}, \\ u(t,x) \geq \xi(t,x)\ dt\times dx\times d\mathbb{P} - a.e., \\ \displaystyle\int_Q (u(t,x)-\xi(t,x))\mu(dt,dx) = 0, \end{cases} \tag{2.1}$$

with general Dirichlet boundary conditions. Here and in what follows, the usual summation convention is applied, ξ is a given stochastic process called the *obstacle process*, defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, $T \in (0,\infty)$ is a deterministic *terminal time*, $\mathcal{O} \subset \mathbb{R}^n$ is a possibly unbounded domain, $\partial_j u = \frac{\partial u}{\partial x_j}$ and $\nabla = (\partial_1, \cdots, \partial_d)$ denotes the gradient operator. A *solution* to the RBSPDE is a random triple (u, v, μ) defined on $\Omega \times [0,T] \times \mathbb{R}^n$ such that (2.1) holds in a suitable sense.

This chapter is organized as follows: in Section 2.1, we list some notations and the standing assumptions on the parameters of the RBSPDE (2.1). The existence and uniqueness of weak solution to the RBSPDE (2.1) with a general Dirichlet boundary condition is presented in Section 2.2. In Section 2.3, we establish the maximum principle for the RBSPDE (2.1) on a general domain as well as the maximum principles for RBSPDEs on a bounded domain and BSPDEs on a general domain. The local behavior of the weak solutions to (2.1) is also considered. Finally, we list in the Appendix A.1 and Appendix A.2 some useful lemmas, the frequently used Itô formulas and some definitions related to the stochastic regular measure, respectively.

2.1. Preliminaries and standing assumptions

For an arbitrary domain Π in some Euclidean space, let $\mathcal{C}_0^\infty(\Pi)$ be the class of infinitely differentiable functions with compact support in Π, and $L^2(\Pi)$ be the usual square integrable space on Π with the scalar product $\langle u, v\rangle_\Pi = \int_\Pi u(x)v(x)dx$ and the norm $\|u\|_{L^2(\Pi)} = \langle u, u\rangle_\Pi^{\frac{1}{2}}$ for each pair $u, v \in L^2(\Pi)$. For $(k, p) \in \mathbb{Z} \times [1, \infty)$ where $\mathbb{Z}$ is the set of all the integers, let $H^{k,p}(\Pi)$ be the usual k-th order Sobolev space. For convenience, when $\Pi = \mathcal{O}$, we write $\langle \cdot, \cdot\rangle$ and $\|\cdot\|$ for $\langle \cdot, \cdot\rangle_\mathcal{O}$ and $\|\cdot\|_{L^2(\mathcal{O})}$ respectively. We recall that $Q = [0, T] \times \mathcal{O}$.

For $t \in [0, T]$ and $\Pi \subseteq \mathbb{R}^n$, we put $\Pi_t := [t, T] \times \Pi$. Denote by $H^{k,p}_\mathcal{F}(\Pi_t)$ the class of $H^{k,p}(\Pi)$-valued predictable processes on $[t, T]$ such that for each $u \in H^{k,p}_\mathcal{F}(\Pi_t)$ we have that

$$\|u\|_{H^{k,p}_\mathcal{F}(\Pi_t)} := \left(\mathbb{E}\left[\int_t^T \|u(s, \cdot)\|^p_{H^{k,p}(\Pi)} ds\right]\right)^{1/p} < \infty.$$

Let $\mathcal{M}^{k,p}(\Pi_t)$ be the subspace of $H^{k,p}_\mathcal{F}(\Pi_t)$ such that

$$\|u\|_{k,p;\Pi_t} := \left(\operatorname{esssup}_{\omega\in\Omega} \sup_{s\in[t,T]} \mathbb{E}\left[\int_s^T \|u(\omega, \tau, \cdot)\|^p_{H^{k,p}_\mathcal{F}(\Pi)} d\tau|\mathcal{F}_s\right]\right)^{1/p} < \infty$$

and $\mathcal{L}^\infty(\Pi_t)$ be the subspace of $H^{0,p}_\mathcal{F}(\Pi_t)$ such that

$$\|u\|_{\infty;\Pi_t} := \operatorname{esssup}_{(\omega,s,x)\in\Omega\times\Pi_t}|u(\omega, s, x)| < \infty.$$

Denote by $\mathcal{L}^{\infty,p}(\Pi_t)$ the subspace of $H^{0,p}_\mathcal{F}(\Pi_t)$ such that

$$\|u\|_{\infty,p;\Pi_t} := \operatorname{esssup}_{(\omega,s)\in\Omega\times[t,T]}\|u(\omega, s, \cdot)\|_{L_p(\Pi)} < \infty.$$

Let $\mathcal{V}_2(\Pi_t)$ be the class of all $u \in H^{1,2}_\mathcal{F}(\Pi_t)$ such that

$$\|u\|_{\mathcal{V}_2(\Pi_t)} := \left(\|u\|^2_{\infty,2;\Pi_t} + \|\nabla u\|^2_{0,2;\Pi_t}\right)^{1/2} < \infty$$

and let $\mathcal{V}_{2,0}(\Pi_t)$ be the subspace of $\mathcal{V}_2(\Pi_t)$ for which

$$\lim_{r\to 0} ||u(s + r, \cdot) - u(s, \cdot)||_{L^2(\Pi)} = 0 \quad \text{for all } s, s + r \in [t, T], \quad \text{a.s.}$$

Assumption 2.1.1. We assume throughout that the coefficients and the obstacle process of the RBSPDE (1.1) satisfy the following conditions. Denote by $\mathbb{F}$ the σ-algebra generated by all predictable sets on $\Omega \times [0, T]$ associated with $(\mathcal{F}_t)_{t\geq 0}$.

$(\mathcal{A}_1)$ The random functions

$$g(\cdot, \cdot, \cdot, X, Y, Z) : \Omega\times[0, T]\times\mathcal{O} \to \mathbb{R}^n \quad \text{and} \quad f(\cdot, \cdot, \cdot, X, Y, Z) : \Omega\times[0, T]\times\mathcal{O} \to \mathbb{R}$$

are $\mathbb{F} \otimes \mathcal{B}(\mathcal{O})$-measurable for any $(X, Y, Z) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$ and there exist positive constants L, κ and β such that for each $(X_i, Y_i, Z_i) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$, $i = 1, 2$,

$$|g(\cdot, \cdot, \cdot, X_1, Y_1, Z_1)-g(\cdot, \cdot, \cdot, X_2, Y_2, Z_2)| \leq L|X_1-X_2|+\frac{\kappa}{2}|Y_1-Y_2|+\sqrt{\beta}|Z_1-Z_2|$$

and

$$|f(\cdot, \cdot, \cdot, X_1, Y_1, Z_1)-f(\cdot, \cdot, \cdot, X_2, Y_2, Z_2)| \leq L(|X_1-X_2|+|Y_1-Y_2|+|Z_1-Z_2|).$$

($\mathcal{A}_2$) The coefficients a and σ are $\mathbb{F}\otimes\mathcal{B}(\mathcal{O})$-measurable and there exist positive constants $\varrho>1$, λ and Λ such that for each $\eta\in\mathbb{R}^n$ and $(\omega,t,x)\in\Omega\times[0,T]\times\mathcal{O}$,

$$\lambda|\eta|^2 \leq (2a^{ij}(\omega,t,x)-\varrho\sigma^{ir}\sigma^{jr}(\omega,t,x))\eta^i\eta^j \leq \Lambda|\eta|^2$$
$$|a(\omega,t,x)|+|\sigma(\omega,t,x)|\leq\Lambda,$$

and

$$\lambda-\kappa-\varrho'\beta>0 \textit{ with } \varrho':=\frac{\varrho}{\varrho-1}.$$

($\mathcal{A}_3$) The terminal value satisfies $G\in L^\infty(\Omega,\mathcal{F}_T,L^2(\mathcal{O}))\cap L^\infty(\Omega,\mathcal{O})$ and for some $p>\max\{n+2,2+4/n\}$, one has

$$g_0:=g(\cdot,\cdot,\cdot,0,0,0)\in\mathcal{M}^{0,p}(Q)\cap\mathcal{M}^{0,2}(Q)$$
$$f_0:=f(\cdot,\cdot,\cdot,0,0,0)\in\mathcal{M}^{0,\frac{p(n+2)}{p+n+2}}(Q)\cap\mathcal{M}^{0,2}(Q).$$

($\mathcal{A}_4$) The obstacle process ξ is almost surely quasi-continuous (see Appendix for the definition) on Q and there exists a process $\hat{\xi}$ such that $\xi\leq\hat{\xi}$ $ds\times dx\times d\mathbb{P}$-a.e., where $\hat{\xi}\in\mathcal{V}_{2,0}(Q)$ together with some $\hat{v}\in\mathcal{M}^{0,2}(Q)$ is a solution to BSPDE

$$\begin{cases} -d\hat{\xi}(t,x)=[\partial_j(a^{ij}\partial_i\hat{\xi}(t,x)+\sigma^{jr}\hat{v}^r(t,x))+\hat{f}(t,x)+\nabla\cdot\hat{g}(t,x)]dt \\ \qquad\qquad -\hat{v}^r(t,x)dW_t^r, \qquad (t,x)\in Q, \\ \hat{\xi}(T,x)=\hat{G}(x), \quad x\in\mathcal{O}, \end{cases} \tag{2.2}$$

with the random functions $\hat{f}$, $\hat{g}$ and $\hat{G}$ satisfying

$$\hat{G}\in L^\infty(\Omega,\mathcal{F}_T,L^2(\mathcal{O}))\cap L^\infty(\Omega,\mathcal{O}),$$
$$\hat{f}\in\mathcal{M}^{0,\frac{p(n+2)}{p+n+2}}(Q)\cap\mathcal{M}^{0,2}(Q),$$
$$\hat{g}\in\mathcal{M}^{0,p}(Q)\cap\mathcal{M}^{0,2}(Q).$$

($\mathcal{A}_5$) The function $x\mapsto g(\cdot,\cdot,\cdot,x,0,0)$ is uniformly Lipschitz continuous in norm:

$$\|g(\cdot,\cdot,\cdot,X_1,0,0)-g(\cdot,\cdot,\cdot,X_2,0,0)\|_{0,p;Q}\leq L|X_1-X_2|;$$
$$\|g(\cdot,\cdot,\cdot,X_1,0,0)-g(\cdot,\cdot,\cdot,X_2,0,0)\|_{0,2;Q}\leq L|X_1-X_2|.$$

Remark 2.1.2. While the assumptions ($\mathcal{A}_1-\mathcal{A}_4$) are standard for the existence and uniqueness of solution, the assumption $\mathcal{A}_5$ is required for the iteration scheme for proof of the maximum principle in Theorem 2.3.1 below, which follows easily from ($\mathcal{A}_1$) when the domain is bounded.

For the index p specified in ($\mathcal{A}_3$) and $t\in[0,T]$, define the functional A_p and B_2 as follows:

$$A_p(l,h;\mathcal{O}_t):=\|l\|_{0,\frac{p(n+2)}{p+n+2};\mathcal{O}_t}+\|h\|_{0,p;\mathcal{O}_t},\quad (l,h)\in\mathcal{M}^{0,\frac{p(n+2)}{p+n+2}}(\mathcal{O}_t)\times\mathcal{M}^{0,p}(\mathcal{O}_t)$$

and

$$B_2(l,h;\mathcal{O}_t) := \|l\|_{0,2;\mathcal{O}_t} + \|h\|_{0,2;\mathcal{O}_t}, \quad (l,h) \in \mathcal{M}^{0,2}(\mathcal{O}_t) \times \mathcal{M}^{0,2}(\mathcal{O}_t).$$

In Sections 2.2 and 2.3, we will repeatedly use the Young inequality of the form

$$\langle f, g\rangle = \langle \sqrt{\epsilon} f, \frac{1}{\sqrt{\epsilon}} g\rangle \leq \frac{1}{2}\left[\epsilon\|f\|^2 + \frac{1}{\epsilon}\|g\|^2\right]. \tag{2.3}$$

2.2. Existence and uniqueness of weak solution to RBSPDE (2.1)

In this section we prove an existence and uniqueness of weak solutions result for the RBSPDE (2.1) along with a strong norm estimate. The difficulty in defining weak solutions to the RBSPDE (2.1) is the random measure μ. It is typically a local time so the Skorokhod condition $\int_Q (u-\xi)\,\mu(dt,dx) = 0$ might not make sense. To give a rigorous meaning to the integral condition, the theory of parabolic potential and capacity introduced by [Pie79, Pie80] was generalized by [QW14] to a backward stochastic framework. We recall the definition of quasi continuity and stochastic regular measures in Appendix A.2.

Definition 2.2.1. The triple (u, v, μ) is called a weak solution to the RBSPDE (2.1) if:

(1) $(u, v) \in \mathcal{V}_{2,0}(Q) \times \mathcal{M}^{0,2}(Q)$ and μ is a stochastic regular measure;

(2) the RBSPDE (2.1) holds in the weak sense, i.e., for each $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^+)\otimes\mathcal{C}_0^\infty(\mathcal{O})$, we have

$$\begin{aligned}
&\langle u(t,\cdot), \varphi(t,\cdot)\rangle \\
=&\langle G(\cdot), \varphi(T,\cdot)\rangle \\
&-\int_t^T \left\{\langle u(s,\cdot), \partial_s\varphi(s,\cdot)\rangle + \langle \partial_j\varphi(s,\cdot), a^{ij}(s,\cdot)\partial_i u(s,\cdot) + \sigma^{jr}v^r(s,\cdot)\rangle\right\} ds \\
&+\int_t^T \left[\langle f(s,\cdot,u(s,\cdot),\nabla u(s,\cdot),v(s,\cdot)), \varphi(s,\cdot)\rangle\right. \\
&\left.-\langle g^j(s,\cdot,u(s,\cdot),\nabla u(s,\cdot),v(s,\cdot)), \partial_j\varphi(s,\cdot)\rangle\right] ds \\
&+\int_{[t,T]\times\mathcal{O}} \varphi(s,x)\mu(ds,dx) - \int_t^T \langle \varphi(s,\cdot), v^r(s,\cdot)dW_s^r\rangle, \quad \text{a.s.};
\end{aligned}$$

(3) u admits a quasi-continuous version $\tilde{u}$ such that $\tilde{u} \geq \xi$ $ds \times dx \times d\mathbb{P}$ *a.e.* and

$$\int_Q (\tilde{u}(t,x) - \xi(t,x))\mu(dt,dx) = 0 \quad \mathbb{P}\text{-a.s.} \tag{2.4}$$

We denote by $\mathcal{U}(\xi, f, g, G)$ the set of all the weak solutions of the RBSPDE (2.1) associated with the obstacle process ξ, the terminal condition G, and the coefficients f and g. Further, $\mathcal{U}(-\infty, f, g, G)$ is the set of solutions when there is no obstacle, i.e., $\mathcal{U}(-\infty, f, g, G)$ is the set of solution pairs (u, v) to the associated BSPDE with terminal condition G and coefficients f and g.

The following theorem guarantees the existence and uniqueness of weak solutions in the sense of Definition 2.2.1. The arguments for the norm estimate also apply to Lemma 2.3.3 below, which is needed for the proof of our maximum principle.

Theorem 2.2.2. *Suppose that Assumptions $(\mathcal{A}_1)$-$(\mathcal{A}_4)$ hold and that $\hat{\xi}|_{\partial\mathcal{O}} = 0$. Then the RBSPDE (1.1) admits a unique solution (u, v, μ) that satisfies the zero Dirichlet condition $u|_{\partial\mathcal{O}} = 0$. Moreover, for each $t \in [0, T]$, one has*

$$\begin{aligned}\|u\|_{\mathcal{V}_2(\mathcal{O}_t)} + \|v\|_{0,2;\mathcal{O}_t} \leq & C \Big(esssup_{\omega\in\Omega} \|G(\omega, \cdot)\|_{L^2(\mathcal{O})} + esssup_{\omega\in\Omega} \|\hat{G}(\omega, \cdot)\|_{L^2(\mathcal{O})} \\ & + B_2(f_0, g_0; \mathcal{O}_t) + B_2(\hat{f}, \hat{g}; \mathcal{O}_t) \Big),\end{aligned} \tag{2.5}$$

where the positive constant C only depends on the constants λ, ϱ, κ, β, L and T.

Proof. It has been shown in [QW14, Theorem 4.12] that the RBSPDE (1.1) admits a unique solution (u, v, μ) satisfying the zero Dirichlet condition $u|_{\partial\mathcal{O}} = 0$ and that this solutions satisfies the integrability condition

$$\mathbb{E}\left[\sup_{t\in[0.T]} \|u(t)\|^2\right] + \mathbb{E}\left[\int_0^T \|\nabla u(t)\|^2 \, dt\right] + \mathbb{E}\left[\int_0^T \|v(t)\|^2 \, dt\right] < \infty.$$

Hence, we only need to prove the estimate (2.5). To this end, notice first that

$$\begin{aligned}&\int_t^T \int_{\mathcal{O}} (u(s, x) - \hat{\xi}(s, x)) \, \mu(dsdx) \\ =& \int_t^T \int_{\mathcal{O}} (u(s, x) - \xi(s, x) + \xi(s, x) - \hat{\xi}(s, x)) \, \mu(dsdx) \\ \leq & 0.\end{aligned}$$

Thus for each $t \in [0,T]$, Proposition A.1.4 yields almost surely,

$$
\begin{aligned}
&\|u(t)-\hat{\xi}(t)\|^2+\int_t^T \|v(s)-\hat{v}(s)\|^2 ds\\
=&\|G-\hat{G}\|^2-\int_t^T \langle u(s)-\hat{\xi}(s), v^r(s)-\hat{v}^r(s)\rangle\, dW_s^r\\
&-\int_t^T \langle 2\partial_j(u-\hat{\xi}(s)), a^{ij}\partial_i(u-\hat{\xi})(s)+\sigma^{jr}(v^r-\hat{v}^r)\rangle\, ds\\
&-\int_t^T \langle 2\partial_j(u-\hat{\xi}(s)), g^j(s,u(s),\nabla u(s),v(s))-\hat{g}^j(s)\rangle\, ds\\
&+\int_t^T \langle 2(u-\hat{\xi}(s)), f(s,u(s),\nabla u(s),v(s))-\hat{f}(s)\rangle\, ds\\
&+\int_{\mathcal{O}_t} 2(u(s,x)-\hat{\xi}(s,x))\mu(ds,dx)\\
\leq&\|G-\hat{G}\|^2-\int_t^T \langle u(s)-\hat{\xi}(s), v^r(s)-\hat{v}^r(s)\rangle\, dW_s^r\\
&-\int_t^T \langle 2\partial_j(u-\hat{\xi}(s)), a^{ij}\partial_i(u-\hat{\xi})(s)+\sigma^{jr}(v^r-\hat{v}^r)\rangle\, ds\\
&-\int_t^T \langle 2\partial_j(u-\hat{\xi}(s)), g^j(s,u(s),\nabla u(s),v(s))-\hat{g}^j(s)\rangle\, ds\\
&+\int_t^T \langle 2(u-\hat{\xi}(s)), f(s,u(s),\nabla u(s),v(s))-\hat{f}(s)\rangle\, ds.
\end{aligned}
\tag{2.6}
$$

Applying assumption $(\mathcal{A}_2)$ and (2.3), one has

$$
\begin{aligned}
I_1:=&-\mathbb{E}\left[\int_t^T \langle 2\partial_j(u-\hat{\xi}(s)), a^{ij}\partial_i(u-\hat{\xi})(s)+\sigma^{jr}(v_r-\hat{v}^r)\rangle\, ds\Big|\mathcal{F}_t\right]\\
=&-\mathbb{E}\left[\int_t^T \langle \partial_j(u-\hat{\xi}(s)), (2a^{ij}-\sigma^{ir}\sigma^{jr}\varrho)\partial_i(u-\hat{\xi})(s)\right.\\
&\quad\left.+\sigma^{ir}\sigma^{jr}\varrho\partial_i(u-\hat{\xi})(s)+2\sigma^{jr}(v^r-\hat{v}^r)\rangle\, ds|\mathcal{F}_t\right]\\
\leq&-\lambda\mathbb{E}\left[\int_t^T \|\nabla(u(s)-\hat{\xi}(s))\|^2\, ds|\mathcal{F}_t\right]+\frac{1}{\varrho}\mathbb{E}\left[\int_t^T \|v(s)-\hat{v}(s)\|^2\, ds|\mathcal{F}_t\right].
\end{aligned}
$$

By assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_3)$ and the estimate (2.3) it holds for each $\epsilon>0$ and $\theta>0$ that:

$$
\begin{aligned}
I_2:=&-\mathbb{E}\left[\int_t^T \langle 2\partial_j(u-\hat{\xi}(s)), g^j(s,u(s),\nabla u(s),v(s))-\hat{g}^j(s)\rangle\, ds|\mathcal{F}_t\right]\\
\leq&\mathbb{E}\left[\int_t^T \langle 2|\nabla(u(s)-\hat{\xi}(s))|, L|u(s)|+\frac{\kappa}{2}|\nabla u(s)|+\sqrt{\beta}|v(s)|\rangle\, ds|\mathcal{F}_t\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\mathbb{E}\left[\int_t^T \langle 2|\nabla(u(s)-\hat{\xi}(s))|, |\hat{g}(s)|+|g_0(s)|\rangle\, ds|\mathcal{F}_t\right]\\
\leq & 2\epsilon\mathbb{E}\left[\int_t^T \|\nabla(u(s)-\hat{\xi}(s))\|^2\, ds|\mathcal{F}_t\right]+C(\epsilon)\mathbb{E}\left[\int_t^T \|g_0(s)\|^2\, ds|\mathcal{F}_t\right]\\
&+C(\epsilon)\mathbb{E}\left[\int_t^T \|\hat{g}(s)\|^2\, ds|\mathcal{F}_t\right]\\
&+\mathbb{E}\left[\int_t^T \langle 2|\nabla(u(s)-\hat{\xi}(s))|, L|u(s)-\hat{\xi}(s)|+L|\hat{\xi}(s)|\rangle\, ds|\mathcal{F}_t\right]\\
&+\mathbb{E}\left[\int_t^T \langle 2|\nabla(u(s)-\hat{\xi}(s))|, \frac{\kappa}{2}|\nabla(u(s)-\hat{\xi}(s))|+\frac{\kappa}{2}|\nabla\hat{\xi}(s)|\rangle\, ds|\mathcal{F}_t\right]\\
&+\mathbb{E}\left[\int_t^T \langle 2|\nabla(u(s)-\hat{\xi}(s))|, \sqrt{\beta}|v(s)-\hat{v}(s)|+\sqrt{\beta}|\hat{v}(s)|\rangle\, ds|\mathcal{F}_t\right]\\
\leq & 2\epsilon\mathbb{E}\left[\int_t^T \|\nabla(u(s)-\hat{\xi}(s))\|^2\, ds|\mathcal{F}_t\right]+C(\epsilon)\mathbb{E}\left[\int_t^T \|\hat{g}(s)\|\, ds|\mathcal{F}_t\right]\\
&+C(\epsilon)\mathbb{E}\left[\int_t^T \|g_0(s)\|\, ds|\mathcal{F}_t\right]+2\epsilon\mathbb{E}\left[\int_t^T \|\nabla(u(s)-\hat{\xi}(s))\|^2\, ds|\mathcal{F}_t\right]\\
&+C(\epsilon,L)\mathbb{E}\left[\int_t^T \|u(s)-\hat{\xi}(s)\|^2\, ds|\mathcal{F}_t\right]+C(\epsilon,L)\mathbb{E}\left[\int_t^T \|\hat{\xi}(s)\|^2\, ds|\mathcal{F}_t\right]\\
&+\kappa\mathbb{E}\left[\int_t^T \|\nabla(u(s)-\hat{\xi}(s))\|^2\, ds|\mathcal{F}_t\right]+\epsilon\mathbb{E}\left[\int_t^T \|\nabla(u(s)-\hat{\xi}(s))\|^2\, ds|\mathcal{F}_t\right]\\
&+C(\epsilon,\kappa)\mathbb{E}\left[\int_t^T \|\nabla\hat{\xi}(s))\|^2\, ds|\mathcal{F}_t\right]+\beta\theta\mathbb{E}\left[\int_t^T \|\nabla(u(s)-\hat{\xi}(s))\|^2\, ds|\mathcal{F}_t\right]\\
&+\frac{1}{\theta}\mathbb{E}\left[\int_t^T \|v(s)-\hat{v}(s)\|^2\, ds|\mathcal{F}_t\right]+\epsilon\mathbb{E}\left[\int_t^T \|\nabla(u(s)-\hat{\xi}(s))\|^2\, ds|\mathcal{F}_t\right]\\
&+C(\epsilon,\beta)\mathbb{E}\left[\int_t^T \|\hat{v}(s)\|^2\, ds|\mathcal{F}_t\right]\\
\leq & (6\epsilon+\kappa+\beta\theta)\,\mathbb{E}\left[\int_t^T \|\nabla(u-\hat{\xi})(s)\|^2\, ds|\mathcal{F}_t\right]+\frac{1}{\theta}\mathbb{E}\left[\int_t^T \|v(s)-\hat{v}(s)\|^2\, ds|\mathcal{F}_t\right]\\
&+C(\epsilon)\mathbb{E}\left[\int_t^T \|g_0(s)\|^2\, ds|\mathcal{F}_t\right]+C(\epsilon)\mathbb{E}\left[\int_t^T \|\hat{g}(s)\|^2\, ds|\mathcal{F}_t\right]\\
&+C(\epsilon,L)\mathbb{E}\left[\int_t^T \|u(s)-\hat{\xi}(s)\|^2\, ds|\mathcal{F}_t\right]+C(\epsilon,L)\mathbb{E}\left[\int_t^T \|\hat{\xi}(s)\|^2\, ds|\mathcal{F}_t\right]
\end{aligned}
$$

$$+ C(\epsilon,\kappa)\mathbb{E}\left[\int_t^T \|\nabla\hat{\xi}(s)\|^2\,ds|\mathcal{F}_t\right] + C(\epsilon,\beta)\mathbb{E}\left[\int_t^T \|\hat{v}(s)\|^2\,ds|\mathcal{F}_t\right].$$

It follows from $(\mathcal{A}_3)$ that:

$$\begin{aligned}
I_3 :&= \mathbb{E}\left[\int_t^T \langle 2(u(s)-\hat{\xi}(s)), f(s,u(s),\nabla u(s),v(s)) - \hat{f}(s)\rangle\,ds|\mathcal{F}_t\right]\\
&\leq \mathbb{E}\left[\int_t^T \langle 2|u(s)-\hat{\xi}(s)|, |f_0| + L|u(s)| + L|\nabla u(s)| + L|v(s)| + |\hat{f}(s)|\rangle\,ds|\mathcal{F}_t\right].
\end{aligned}$$

In view of (2.3) it further holds for each $\epsilon_1 > 0$ that:

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T \langle 2|u(s)-\hat{\xi}(s)|, L|u(s)| + L|\nabla u(s)| + L|v(s)|\rangle\,ds|\mathcal{F}_t\right]\\
\leq&\epsilon_1\mathbb{E}\left[\int_t^T \|\nabla u(s)\|^2\,ds|\mathcal{F}_t\right] + \epsilon_1\mathbb{E}\left[\int_t^T \|v(s)\|^2\,ds|\mathcal{F}_t\right]\\
&+ C(\epsilon_1,L)\mathbb{E}\left[\int_t^T \|u(s)-\hat{\xi}(s)\|^2\,ds|\mathcal{F}_t\right]\\
&+ \mathbb{E}\left[\int_t^T \langle 2|(u-\hat{\xi})(s)|, L|(u-\hat{\xi})(s)| + L|\hat{\xi}(s)|\rangle\,ds|\mathcal{F}_t\right]\\
\leq&2\epsilon_1\mathbb{E}\left[\int_t^T \|\nabla(u(s)-\hat{\xi}(s))\|^2\,ds|\mathcal{F}_t\right] + 2\epsilon_1\mathbb{E}\left[\int_t^T \|v(s)-\hat{v}(s)\|^2 ds|\mathcal{F}_t\right]\\
&+ 2\epsilon_1\mathbb{E}\left[\int_t^T \|\hat{v}(s)\|^2\,ds|\mathcal{F}_t\right] + 2\epsilon_1\mathbb{E}\left[\int_t^T \|\nabla\hat{\xi}(s)\|^2\,ds|\mathcal{F}_t\right]\\
&+ C(\epsilon_1,L)\mathbb{E}\left[\int_t^T \|u(s)-\hat{\xi}(s)\|^2\,ds|\mathcal{F}_t\right] + \mathbb{E}\left[\int_t^T \|\hat{\xi}(s)\|^2\,ds|\mathcal{F}_t\right],
\end{aligned}$$

and by the Hölder inequality one has that:

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T \langle 2|u(s)-\hat{\xi}(s)|, |f_0| + |\hat{f}(s)|\rangle\,ds|\mathcal{F}_t\right]\\
&\leq 2\mathbb{E}\left[\int_t^T \|(u-\hat{\xi})(s)\|^2\,ds|\mathcal{F}_t\right] + \mathbb{E}\left[\int_t^T \|f_0(s)\|^2\,ds|\mathcal{F}_t\right] + \mathbb{E}\left[\int_t^T \|\hat{f}(s)\|^2\,ds|\mathcal{F}_t\right]
\end{aligned}$$

In addition,

$$I_4 := \mathbb{E}\left[\|G-\hat{G}\|^2|\mathcal{F}_t\right] \leq \operatorname{esssup}_{\omega\in\Omega}\|G-\hat{G}\|^2.$$

Summing up the estimates I_1-I_4 and taking the supremum w.r.t. $(\omega,s)\in\Omega\times[t,T]$ on both sides we arrive at:

$$\|u-\hat{\xi}\|^2_{\infty,2;\mathcal{O}_t} + \|v-\hat{v}\|^2_{0,2;\mathcal{O}_t} + (\lambda-\kappa-\beta\theta-6\epsilon-2\epsilon_1)\,\|\nabla(u-\hat{\xi})\|^2_{0,2;\mathcal{O}_t}$$

$$
\begin{aligned}
\leq &\left(\frac{1}{\varrho}+\frac{1}{\theta}+2\epsilon_1\right)\|v-\hat{v}\|^2_{0,2;\mathcal{O}_t}+C(\epsilon,\epsilon_1,L)\int_t^T\|u-\hat{\xi}\|^2_{\infty,2;\mathcal{O}_s}\,ds+C(\epsilon,\epsilon_1,\beta)\|\hat{v}\|^2_{0,2;\mathcal{O}_t}\\
&+C(\epsilon,\epsilon_1,\kappa,L)\|\hat{\xi}\|^2_{\mathcal{V}_2(\mathcal{O}_t)}+\|f_0\|^2_{0,2;\mathcal{O}_t}+\|\hat{f}\|^2_{0,2;\mathcal{O}_t}+C(\epsilon)\left(\|g_0\|^2_{0,2;\mathcal{O}_t}+\|\hat{g}\|^2_{0,2;\mathcal{O}_t}\right)\\
&+\operatorname{esssup}_{\omega\in\Omega}\|G-\hat{G}\|^2.
\end{aligned}
$$

By assumption $(\mathcal{A}_2)$ we can choose $\theta>\varrho'$ such that $\lambda-\kappa-\beta\theta>0$, and $\theta>\varrho'$ also implies $\frac{1}{\varrho}+\frac{1}{\theta}<1$. Now taking ϵ and ϵ_1 small enough such that $\lambda-\kappa-\beta\theta-6\epsilon-2\epsilon_1>0$ and $\frac{1}{\varrho}+\frac{1}{\theta}+2\epsilon_1<1$, we have

$$
\begin{aligned}
&\|u-\hat{\xi}\|^2_{\mathcal{V}^2(\mathcal{O}_t)}+\|v-\hat{v}\|^2_{0,2;\mathcal{O}_t}\\
\leq &C(\epsilon,\epsilon_1,\lambda,\beta,\kappa,L,\varrho)\left(\int_t^T\|u-\hat{\xi}\|^2_{\infty,2;\mathcal{O}_s}\,ds+\|\hat{v}\|^2_{0,2;\mathcal{O}_t}+\|\hat{\xi}\|^2_{\mathcal{V}_2(\mathcal{O}_t)}\right.\\
&\left.+B_2(f_0,g_0;\mathcal{O}_t)^2+B_2(\hat{f},\hat{g};\mathcal{O}_t)^2+\operatorname{esssup}_{\omega\in\Omega}\|G-\hat{G}\|^2_{L^2(\mathcal{O})}\right).
\end{aligned}
\tag{2.7}
$$

By Gronwall's inequality,

$$
\begin{aligned}
&\|u-\hat{\xi}\|^2_{\mathcal{V}_2(\mathcal{O}_t)}+\|v-\hat{v}\|^2_{0,2;\mathcal{O}_t}\\
\leq &C(\epsilon,\epsilon_1,\lambda,\beta,\kappa,L,\varrho,T)\left(\|\hat{v}\|^2_{0,2;\mathcal{O}_t}+\|\hat{\xi}\|^2_{\mathcal{V}_2(\mathcal{O}_t)}+\operatorname{esssup}_{\omega\in\Omega}\|G-\hat{G}\|^2_{L^2(\mathcal{O}_t)}\right.\\
&\left.+B_2(f_0,g_0;\mathcal{O}_t)^2+B_2(\hat{f},\hat{g};\mathcal{O}_t)^2\right).
\end{aligned}
\tag{2.8}
$$

Since $\hat{\xi}|_{\partial\mathcal{O}}=0$, we can apply Proposition A.1.4 to $\|\hat{\xi}(t)\|^2$. Starting from (2.6), using similar estimates,

$$
\|\hat{\xi}\|^2_{\mathcal{V}_2(\mathcal{O}_t)}+\|\hat{v}\|^2_{0,2;\mathcal{O}_t}\leq C\left(B_2(\hat{f},\hat{g};\mathcal{O}_t)^2+\operatorname{esssup}_{\omega\in\Omega}\|\hat{G}\|^2\right),
\tag{2.9}
$$

where C only depends on λ, β, κ, ϱ, L and T. The estimate (2.8) together with (2.9) yields (2.5). $\square$

With the same notation as in Theorem 2.2.2, we can relax the zero Dirichlet boundary condition in Theorem 2.2.2 by assuming $u|_{\partial\mathcal{O}}=\tilde{u}|_{\partial\mathcal{O}}$ for some $(\tilde{u},\tilde{v})\in\mathcal{U}(-\infty,\tilde{f},\tilde{g},\tilde{G})$ where the coefficients a, σ, $\tilde{f}$, $\tilde{g}$ and $\tilde{G}$ satisfy $(\mathcal{A}_2)$ and $(\mathcal{A}_3)$ respectively, and $\tilde{f}$ and $\tilde{g}$ do not depend on $\tilde{u}$, $\nabla\tilde{u}$ and $\tilde{v}$. Assume further that $\hat{\xi}|_{\partial\mathcal{O}}\leq\tilde{u}|_{\partial\mathcal{O}}$ and put $\bar{\xi}:=\hat{\xi}-\tilde{u}$. Then, $(\bar{\xi},\bar{v})\in\mathcal{U}(-\infty,\bar{f},\bar{g},\bar{G})$, where $\bar{v}=\hat{v}-\tilde{v}$, $\bar{f}=\hat{f}-\tilde{f}$, $\bar{g}=\hat{g}-\tilde{g}$ and $\bar{G}=\hat{G}-\tilde{G}$. Suppose now that $(\breve{\xi},\breve{v})\in\mathcal{U}(-\infty,\bar{f},\bar{g},\bar{G})$ with $\breve{\xi}|_{\partial_\mathcal{O}}=0$. Then, $\breve{\xi}|_{\partial\mathcal{O}}=0\geq(\hat{\xi}-\tilde{u})|_{\partial\mathcal{O}}=\bar{\xi}|_{\partial\mathcal{O}}$ and the maximum principle in Lemma 2.3.4 yields $\breve{\xi}\geq\hat{\xi}-\tilde{u}\geq\xi-\tilde{u}$. Therefore, our RBSPDE (2.1) is equivalent to the following one but with zero-Dirichlet condition:

$$\begin{cases} -d\breve{u}(t,x) = \big[\partial_j(a^{ij}\partial_i\breve{u} + \sigma^{jr}\breve{v}^r)(t,x) + (f+\nabla\cdot g)(t,x,\breve{u}+\tilde{u},\nabla(\breve{u}+\tilde{u}),\breve{v}+\tilde{v}) \\ \qquad\qquad -(\tilde{f}+\nabla\cdot\tilde{g})(t,x)\big]\,dt + \mu(dt,x) - \breve{v}^r(t,x)dW_t^r, \quad (t,x)\in Q; \\ \breve{u}(T,x) = G(x) - \tilde{G}(x), \quad x\in\mathcal{O}; \\ \breve{u} \geq \xi - \tilde{u}, \quad \mathbb{P}\otimes dt\otimes dx\text{-a.e.}; \\ \int_Q (\breve{u} - (\xi-\tilde{u}))(t,x)\,\mu(dt,dx) = 0. \end{cases} \tag{2.10}$$

By Theorem 2.2.2, there is a unique solution $u-\tilde{u}$ to the RBSPDE (2.10) satisfying zero-Dirichlet condition. In this way, Theorem 2.2.2 extends to RBSPDEs with general Dirichlet conditions.

2.3. Maximum Principle for RBSPDE

In this section we state and prove our maximum principles for RBSPDEs. We start with a global maximum principle on general domains, which states that the weak solution u is bounded on the whole domain if it is bounded on the parabolic boundary. Subsequently we analyze the local behavior of $u^{\pm}$ when u is not necessarily bounded on the parabolic boundary.

2.3.1. Global Case

This section establishes a maximum principle for the RBSPDE (2.1) on a general domain $\mathcal{O}$. Since the Lebesgue measure of $\mathcal{O}$ might not be bounded, the scheme in [QT12] cannot be applied. Instead, motivated by [Qiu15], we use a stochastic De Girogi's scheme that is independent of the measure of the domain. In what follows $\partial_p Q = (\{T\}\times\mathcal{O})\cup([0,T]\times\partial\mathcal{O})$ denotes the parabolic boundary of Q.

Theorem 2.3.1. *(1) Assume that $(\mathcal{A}_1)$-$(\mathcal{A}_5)$ hold. If the triplet (u,v,μ) is a solution to the RBSPDE (2.1), then*

$$\begin{aligned} &esssup_{(\omega,t,x)\in\Omega\times Q} u^{\pm} \\ \leq &C\Big(esssup_{(\omega,t,x)\in\Omega\times\partial_p Q} u^{\pm} + esssup_{(\omega,t,x)\in\Omega\times\partial_p Q}\hat{\xi}^{\pm} \\ &\quad + A_p(f_0^{\pm}, g_0; Q) + B_2(f_0^{\pm}, g_0; Q) + A_p(\hat{f}^{\pm}, \hat{g}; Q) + B_2(\hat{f}^{\pm}, \hat{g}; Q)\Big), \end{aligned}$$

where the constant C depends only on λ, κ, β, L, ϱ, T, p and n.

(2) If all conditions in (1) hold except assumption $(\mathcal{A}_5)$ is changed to

$$g(t,x,r,0,0) = g(t,x,0,0) \text{ and } f(t,x,r,0,0) \text{ is non-increasing w.r.t. } r, \tag{2.11}$$

then

$$
\begin{aligned}
&esssup_{(\omega,t,x)\in\Omega\times Q} u^{\pm} \\
\leq &esssup_{(\omega,t,x)\in\Omega\times\partial_p Q} u^{\pm} + esssup_{(\omega,t,x)\in\Omega\times\partial_p Q} \hat{\xi}^{\pm} \\
&+ C\left(A_p(f_0^{\pm}, g_0; Q)^{\frac{np}{np+2(p-n-2)}} B_2(f_0^{\pm}, g_0; Q)^{\frac{2(p-n-2)}{np+2(p-n-2)}} \right. \\
&\left. + A_p(\hat{f}^{\pm}, \hat{g}; Q)^{\frac{np}{np+2(p-n-2)}} B_2(\hat{f}^{\pm}, \hat{g}; Q)^{\frac{2(p-n-2)}{np+2(p-n-2)}} \right),
\end{aligned}
$$

where C depends on λ, κ, β, L, ϱ, n, p and T.

Proof. We only consider the estimate for the positive part u^+. The one for the negative part u^- follows analogously. Further, we may w.l.o.g. assume that $f(t,x,r,0,0)$ is non-increasing in r. Otherwise, the desired maximum principle can be derived from the maximum principle for the RBSPDE

$$
\begin{cases}
-d\bar{u}(t,x) = \left[\partial_j(a^{ij}(t,x)\partial_i \bar{u}(t,x) + \sigma^{jr}(t,x)\bar{v}^r(t,x))\right. \\
\qquad\qquad + \bar{f}(t,x,\bar{u}(t,x),\nabla\bar{u}(t,x),\bar{v}(t,x)) \\
\qquad\qquad \left.+\nabla\cdot\bar{g}(t,x,\bar{u}(t,x),\nabla\bar{u}(t,x),\bar{v}(t,x))\right]\,dt \\
\qquad\qquad + \bar{\mu}(dt,x) - \bar{v}^r(t,x)\,dW_t^r, \\
\bar{u}(T,x) = \bar{G}(x), \\
\bar{u}(t,x) \geq \bar{\xi}(t,x) \ dt\times dx\times d\mathbb{P}-a.e., \\
\int_Q (\bar{u}(t,x) - \bar{\xi}(t,x))\,\bar{\mu}(dt,dx) = 0,
\end{cases}
$$

where $\bar{u}(t,x) = e^{Lt}u(t,x)$, $\bar{v}(t,x) = e^{Lt}v(t,x)$, $\bar{\mu}(dt,dx) = e^{Lt}\mu(dt,dx)$, $\bar{G}(x) = e^{LT}G(x)$, $\bar{\xi}(t,x) = e^{Lt}\xi(t,x)$ and

$$
\begin{aligned}
&\bar{f}(t,x,\bar{u}(t,x),\nabla\bar{u}(t,x),\bar{v}(t,x)) \\
= &e^{Lt}f(t,x,e^{-Lt}\bar{u}(t,x),e^{-Lt}\nabla\bar{u}(t,x),e^{-Lt}\bar{v}(t,x)) - L\bar{u}(t,x) \\
&\bar{g}(t,x,\bar{u}(t,x),\nabla\bar{u}(t,x),\bar{v}(t,x)) = e^{Lt}g(t,x,e^{-Lt}\bar{u}(t,x),e^{-Lt}\nabla\bar{u}(t,x),e^{-Lt}\bar{v}(t,x)).
\end{aligned}
$$

Now, for $t\in[0,T]$ define

$$
\bar{k} = \mathrm{esssup}_{(\omega,t,x)\in\Omega\times\partial_p\mathcal{O}_t} u^+ + \mathrm{esssup}_{(\omega,t,x)\in\Omega\times\mathcal{O}_t} \hat{\xi}^+.
$$

For a positive constant k to be determined later and each $m\in\mathbb{N}_0$, let $\bar{k}_m =$

$k(1-2^{-m})$ and $k_m = \bar{k}_m + \bar{k}$. By Theorem A.1.5, for $m \geq 1$,

$$\begin{aligned}
&\|(u-k_m)^+(t)\|^2 + \int_t^T \|v^{k_m}(s)\|^2 ds \\
=& -2\int_t^T \langle \partial_j (u-k_m)^+(s), a^{ij}\partial_i (u-k_m)^+(s) + \sigma^{jr}(s) v^{k_m,r}(s)\rangle \, ds \\
&-2\int_t^T \langle \partial_j (u-k_m)^+(s), g^{j,k_m}\left(s, (u-k_m)^+(s), \nabla u(s), v^{k_m}(s)\right)\rangle \, ds \\
&+2\int_t^T \langle (u-k_m)^+(s), f^{k_m}\left(s, (u-k_m)^+(s), \nabla u(s), v^{k_m}(s)\right)\rangle \, ds \\
&+2\int_{\mathcal{O}_t} (u-k_m)^+(s,x)\, \mu(ds,dx) - 2\int_t^T \langle (u-k_m)^+(s), v^{r,k_m}(s)\, dW_s^r\rangle,
\end{aligned} \tag{2.12}$$

where $v^{r,k_m} := v^r 1_{\{u>k_m\}}$, $f^{k_m}(\cdot,\cdot,\cdot,X,\cdot,\cdot) := f(\cdot,\cdot,\cdot,X+k_m,\cdot,\cdot)$, $g^{j,k_m}(\cdot,\cdot,\cdot,X,\cdot,\cdot) := g^j(\cdot,\cdot,\cdot,X+k_m,\cdot,\cdot)$. All terms in (2.12) are well defined. In particular, the stochastic integral is in fact a martingale. Taking conditional expectations on both sides w.r.t. $\mathcal{F}_t$ yields the following estimates for the remaining terms. Similar estimates as for I_1 in the proof of Theorem 2.2.2 yield,

$$\begin{aligned}
J_1 :=& -2\mathbb{E}\left[\int_t^T \langle \partial_j (u-k_m)^+(s), a^{ij}\partial_i (u-k_m)^+(s) + \sigma^{jr}(s) v^{r,k_m}(s)\rangle \, ds|\mathcal{F}_t\right] \\
\leq& -\lambda \mathbb{E}\left[\int_t^T \|\nabla (u-k_m)^+(s)\|^2 \, ds|\mathcal{F}_t\right] + \frac{1}{\varrho}\mathbb{E}\left[\int_t^T \|v^{k_m}(s)\|^2 \, ds|\mathcal{F}_t\right].
\end{aligned} \tag{2.13}$$

By analogy to the estimate of I_2, for each $\epsilon > 0$ and $\theta > 0$, we have that

$$\begin{aligned}
J_2 :=& -2\mathbb{E}\left[\int_t^T \langle \partial_j (u-k_m)^+(s), g^{j,k_m}\left(s, (u-k_m)^+(s), \nabla u(s), v^{k_m}(s)\right)\rangle \, ds|\mathcal{F}_t\right] \\
\leq& 2\mathbb{E}\left[\int_t^T \langle |\nabla (u-k_m)^+(s)|, |g_0^{k_m}| + L|(u-k_m)^+(s)| + \frac{\kappa}{2}|\nabla (u-k_m)^+(s)| \right. \\
&\left. + \sqrt{\beta}|v^{k_m}(s)|\rangle \, ds|\mathcal{F}_t\right] \\
\leq& (\kappa + \beta\theta + \epsilon)\mathbb{E}\left[\int_t^T \|\nabla (u-k_m)^+(s)\|^2 \, ds|\mathcal{F}_t\right] + \frac{1}{\theta}\mathbb{E}\left[\int_t^T \|v^{k_m}(s)\|^2 \, ds|\mathcal{F}_t\right] \\
&+ \frac{L^2}{\epsilon}\mathbb{E}\left[\int_t^T \|(u-k_m)^+(s)\|^2 \, ds|\mathcal{F}_t\right] \\
&+ 2\mathbb{E}\left[\int_t^T \langle |\nabla (u-k_m)^+(s)|, |g_0^{k_m}(s)|\rangle \, ds|\mathcal{F}_t\right].
\end{aligned} \tag{2.14}$$

From

$$\left[(u-k_{m-1})^+ - (u-k_m)^+\right] 1_{\{u>k_m\}} = (k_m - k_{m-1})1_{\{u>k_m\}} = 2^{-m}k1_{\{u>k_m\}}$$

we get

$$1_{\{u>k_m\}} \leq \frac{2^m(u-k_{m-1})^+}{k}1_{\{u>k_m\}} \leq \frac{2^m(u-k_{m-1})^+}{k}. \tag{2.15}$$

By (2.15) and ($\mathcal{A}_5$) it holds that:

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T \langle|\nabla(u-k_m)^+(s)|, |g_0^{k_m}(s)|\rangle\, ds|\mathcal{F}_t\right] \\
\leq &\left(\mathbb{E}\left[\int_t^T \|\nabla(u-k_m)^+(s)\|^2\, ds|\mathcal{F}_t\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[\int_t^T \int_{\mathcal{O}} |g_0^{k_m}(s,x)1_{\{u>k_m\}}|^2\, dxds|\mathcal{F}_t\right]\right)^{\frac{1}{2}} \\
\leq &\|\nabla(u-k_m)^+\|_{0,2;\mathcal{O}_t}\|g_0^{k_m}\|_{0,p;\mathcal{O}_t} \left(\mathbb{E}\left[\int_t^T \int_{\mathcal{O}} 1_{\{u>k_m\}}\, dxds|\mathcal{F}_t\right]\right)^{\frac{1}{2}-\frac{1}{p}} \\
\leq &\|\nabla(u-k_m)^+\|_{0,2;\mathcal{O}_t}\|g_0^{k_m}\|_{0,p;\mathcal{O}_t} \left(\mathbb{E}\left[\int_t^T \int_{\mathcal{O}} \left(\frac{2^m(u-k_{m-1})^+}{k}\right)^{\frac{2(n+2)}{n}}\, dxds|\mathcal{F}_t\right]\right)^{\frac{1}{2}-\frac{1}{p}} \\
\leq &\left(\frac{2^m}{k}\right)^{1+\frac{2(p-n-2)}{np}} \|\nabla(u-k_m)^+\|_{0,2;\mathcal{O}_t}\|g_0^{k_m}\|_{0,p;\mathcal{O}_t}\|(u-k_{m-1})^+\|^{1+\frac{2(p-n-2)}{np}}_{0,\frac{2(n+2)}{n};\mathcal{O}_t} \\
\leq &\left(\frac{2^m}{k}\right)^{1+\frac{2(p-n-2)}{np}} \|\nabla(u-k_{m-1})^+\|_{0,2;\mathcal{O}_t}\|g_0^{k_m}\|_{0,p;\mathcal{O}_t}\|(u-k_{m-1})^+\|^{1+\frac{2(p-n-2)}{np}}_{0,\frac{2(n+2)}{n};\mathcal{O}_t} \\
\leq &\left(\frac{2^m}{k}\right)^{1+\frac{2(p-n-2)}{np}} \|\nabla(u-k_{m-1})^+\|_{0,2;\mathcal{O}_t}(\|g_0\|_{0,p;\mathcal{O}_t} + Lk_m)\|(u-k_{m-1})^+\|^{1+\frac{2(p-n-2)}{np}}_{0,\frac{2(n+2)}{n};\mathcal{O}_t}.
\end{aligned} \tag{2.16}$$

Combining (2.14) and (2.16), we see that

$$\begin{aligned}
J_2 \leq &(\kappa+\beta\theta+\epsilon)\mathbb{E}\left[\int_t^T \|\nabla(u-k_m)^+(s)\|^2\, ds|\mathcal{F}_t\right] + \frac{1}{\theta}\mathbb{E}\left[\int_t^T \|v^{k_m}(s)\|^2\, ds|\mathcal{F}_t\right] \\
&+ \frac{L^2}{\epsilon}\mathbb{E}\left[\int_t^T \|(u-k_m)^+(s)\|^2\, ds|\mathcal{F}_t\right] \\
&+ 2\left(\frac{2^m}{k}\right)^{1+\frac{2(p-n-2)}{np}} \|\nabla(u-k_{m-1})^+\|_{0,2;\mathcal{O}_t}(\|g_0\|_{0,p;\mathcal{O}_t} + Lk_m) \\
&\times \|(u-k_{m-1})^+\|^{1+\frac{2(p-n-2)}{np}}_{0,\frac{2(n+2)}{n};\mathcal{O}_t}.
\end{aligned} \tag{2.17}$$

For each $\epsilon_1 > 0$, by (2.3) the monotonicity of $f(t, x, r, 0, 0)$ yields:

$$
\begin{aligned}
J_3 :=& 2\mathbb{E}\left[\int_t^T \langle (u-k_m)^+(s), f^{k_m}\left(s, (u-k_m)^+(s), \nabla u(s), v^{k_m}(s)\right)\rangle \, ds|\mathcal{F}_t\right] \\
\leq & 2\mathbb{E}\left[\int_t^T \langle (u-k_m)^+(s), f_0^{k_m}(s) + L(u-k_m)^+(s) \right. \\
& \left. + L\nabla(u-k_m)^+(s) + L|v^{k_m}(s)|\rangle \, ds|\mathcal{F}_t\right] \\
\leq & 2\mathbb{E}\left[\int_t^T \langle (u-k_m)^+(s), f_0(s) + L(u-k_m)^+(s) \right. \\
& \left. + L\nabla(u-k_m)^+(s) + L|v^{k_m}(s)|\rangle \, ds|\mathcal{F}_t\right] \\
\leq & \left(2L + \frac{2L^2}{\epsilon_1}\right)\mathbb{E}\left[\int_t^T \|(u-k_m)^+(s)\|^2 \, ds|\mathcal{F}_t\right] \\
& + \epsilon_1\mathbb{E}\left[\int_t^T \|\nabla(u-k_m)^+(s)\|^2 \, ds|\mathcal{F}_t\right] + \epsilon_1\mathbb{E}\left[\int_t^T \|v^{k_m}(s)\|^2 \, ds|\mathcal{F}_t\right] \\
& + 2\mathbb{E}\left[\int_t^T \langle (u-k_m)^+(s), f_0(s)\rangle \, ds|\mathcal{F}_t\right].
\end{aligned}
\tag{2.18}
$$

By (2.15) again, we have

$$
\begin{aligned}
& \mathbb{E}\left[\int_t^T \langle (u-k_m)^+(s), f_0(s)\rangle \, ds|\mathcal{F}_t\right] \\
\leq & \mathbb{E}\left[\int_t^T \langle (u-k_m)^+(s), f_0^+(s)\rangle \, ds|\mathcal{F}_t\right] \\
\leq & \left(\mathbb{E}\left[\int_t^T \int_{\mathcal{O}} |(u-k_m)^+|^{\frac{2(n+2)}{n}} \, dxds|\mathcal{F}_t\right]\right)^{\frac{n}{2(n+2)}} \\
& \times \left(\mathbb{E}\left[\int_t^T \int_{\mathcal{O}} |f_0^+(s,x)|^{\frac{p(n+2)}{p+n+2}} \, dxds|\mathcal{F}_t\right]\right)^{\frac{p+n+2}{p(n+2)}} \\
& \times \left(\mathbb{E}\left[\int_t^T \int_{\mathcal{O}} 1_{\{u>k_m\}} \, dxds|\mathcal{F}_t\right]\right)^{\frac{1}{2}-\frac{1}{p}} \\
\leq & \|(u-k_m)^+\|_{0,\frac{2(n+2)}{n};\mathcal{O}_t} \|f_0^+\|_{0,\frac{p(n+2)}{p+n+2};\mathcal{O}_t} \\
& \times \left(\mathbb{E}\left[\int_t^T \int_{\mathcal{O}} \left(\frac{2^m(u-k_{m-1})^+}{k}\right)^{\frac{2(n+2)}{n}} \, dxds|\mathcal{F}_t\right]\right)^{\frac{1}{2}-\frac{1}{p}}
\end{aligned}
$$

$$\le \left(\frac{2^m}{k}\right)^{1+\frac{2(p-n-2)}{np}} \|(u-k_m)^+\|_{0,\frac{2(n+2)}{n};\mathcal{O}_t} \|(u-k_{m-1})^+\|^{1+\frac{2(p-n-2)}{np}}_{0,\frac{2(n+2)}{n};\mathcal{O}_t} \|f_0^+\|_{0,\frac{p(n+2)}{p+n+2};\mathcal{O}_t}$$

$$\le \left(\frac{2^m}{k}\right)^{1+\frac{2(p-n-2)}{np}} \|(u-k_{m-1})^+\|^{2+\frac{2(p-n-2)}{np}}_{0,\frac{2(n+2)}{n};\mathcal{O}_t} \|f_0^+\|_{0,\frac{p(n+2)}{p+n+2};\mathcal{O}_t}. \tag{2.19}$$

Therefore, by (2.18) and (2.19) we conclude

$$\begin{aligned} J_3 \le & \left(2L + \frac{2L^2}{\epsilon_1}\right) \mathbb{E}\left[\int_t^T \|(u-k_m)^+(s)\|^2\,ds|\mathcal{F}_t\right] \\ & + \epsilon_1 \mathbb{E}\left[\int_t^T \|\nabla(u-k_m)^+(s)\|^2\,ds|\mathcal{F}_t\right] + \epsilon_1 \mathbb{E}\left[\int_t^T \|v^{k_m}(s)\|^2\,ds|\mathcal{F}_t\right] \\ & + 2\left(\frac{2^m}{k}\right)^{1+\frac{2(p-n-2)}{np}} \|(u-k_{m-1})^+\|^{2+\frac{2(p-n-2)}{np}}_{0,\frac{2(n+2)}{n};\mathcal{O}_t} \|f_0^+\|_{0,\frac{p(n+2)}{p+n+2};\mathcal{O}_t}. \end{aligned} \tag{2.20}$$

Finally, note that

$$\int_t^T \int_{\mathcal{O}} (u-k_m)^+\,\mu(dxds) \le \int_t^T \int_{\mathcal{O}} (u-\xi)^+\,\mu(dxds) + \int_t^T \int_{\mathcal{O}} (\xi - \hat{\xi}^+)^+\,\mu(dxds) = 0.$$

Combining the above estimates, we get

$$\begin{aligned} & \|(u-k_m)^+(t)\|^2 + \mathbb{E}\left[\int_t^T \|v^{k_m}(s)\|^2\,ds|\mathcal{F}_t\right] \\ \le & (-\lambda + \kappa + \beta\theta + \epsilon + \epsilon_1)\mathbb{E}\left[\int_t^T \|\nabla(u-k_m)^+\|^2\,ds|\mathcal{F}_t\right] \\ & + \left(\frac{1}{\theta} + \frac{1}{\varrho} + \epsilon_1\right)\mathbb{E}\left[\int_t^T \|v^{k_m}\|^2\,ds|\mathcal{F}_t\right] \\ & + \left(2L + \frac{L^2}{\epsilon} + \frac{2L^2}{\epsilon_1}\right)\mathbb{E}\left[\int_t^T \|(u-k_m)^+\|^2\,ds|\mathcal{F}_t\right] \\ & + 2\left(\frac{2^m}{k}\right)^{1+\frac{2(p-n-2)}{np}} \|\nabla(u-k_{m-1})^+\|_{0,2;\mathcal{O}_t} \\ & \times \|(u-k_{m-1})^+\|^{1+\frac{2(p-n-2)}{np}}_{0,\frac{2(n+2)}{n};\mathcal{O}_t} (\|g_0\|_{0,p;\mathcal{O}_t} + Lk_m) \\ & + 2\left(\frac{2^m}{k}\right)^{1+\frac{2(p-n-2)}{np}} \|(u-k_{m-1})^+\|^{2+\frac{2(p-n-2)}{np}}_{0,\frac{2(n+2)}{n};\mathcal{O}_t} \|f_0^+\|_{0,\frac{p(n+2)}{p+n+2};\mathcal{O}_t}. \end{aligned}$$

From this it is straightforward to see that

$$\min\{1, \lambda - \kappa - \beta\theta - \epsilon - \epsilon_1\}\left\{\|(u-k_m)^+(t)\|^2 + \mathbb{E}\left[\int_t^T \|\nabla(u-k_m)^+\|^2\,ds|\mathcal{F}_t\right]\right\}$$

$$
\begin{aligned}
&+\left(1-\frac{1}{\theta}-\frac{1}{\varrho}-\epsilon_1\right)\mathbb{E}\left[\int_t^T \|v^{k_m}\|^2\,ds|\mathcal{F}_t\right]\\
\le&\left(2L+\frac{L^2}{\epsilon}+\frac{2L^2}{\epsilon_1}\right)\mathbb{E}\left[\int_t^T \|(u-k_m)^+\|^2\,ds|\mathcal{F}_t\right]\\
&+2\left(\frac{2^m}{k}\right)^{1+\frac{2(p-n-2)}{np}}\|\nabla(u-k_{m-1})^+\|_{0,2;\mathcal{O}_t}\\
&\times\|(u-k_{m-1})^+\|_{0,\frac{2(n+2)}{n};\mathcal{O}_t}^{1+\frac{2(p-n-2)}{np}}\left(\|g_0\|_{0,p;\mathcal{O}_t}+Lk_m\right)\\
&+2\left(\frac{2^m}{k}\right)^{1+\frac{2(p-n-2)}{np}}\|(u-k_{m-1})^+\|_{0,\frac{2(n+2)}{n};\mathcal{O}_t}^{2+\frac{2(p-n-2)}{np}}\|f_0^+\|_{0,\frac{p(n+2)}{p+n+2};\mathcal{O}_t}.
\end{aligned}
$$

By assumption $(\mathcal{A}_2)$, there exists $\theta>\varrho'$ such that $\lambda-\kappa-\theta\beta>0$ and $\frac{1}{\theta}+\frac{1}{\varrho}<1$. So, we can take ϵ and ϵ_1 small enough such that $\lambda-\kappa-\beta\theta-\epsilon-\epsilon_1>0$ and $1-\frac{1}{\theta}-\frac{1}{\varrho}-\epsilon_1>0$. Taking the supremum on both sides, Lemma A.1.3 yields,

$$
\begin{aligned}
&\|(u-k_m)^+\|_{\mathcal{V}_2(\mathcal{O}_t)}^2+\|v^{k_m}\|_{0,2;\mathcal{O}_t}^2\\
\le&C_1(\lambda,\kappa,\beta,L,\theta,\varrho,\epsilon,\epsilon_1)\int_t^T\|(u-k_m)^+\|_{\mathcal{V}_2(\mathcal{O}_s)}^2\,ds\\
&+C_1(\lambda,\kappa,\beta,L,\theta,\varrho,n,\epsilon,\epsilon_1)\left(\frac{2^m}{k}\right)^{1+\frac{2(p-n-2)}{np}}\\
&\times\|(u-k_{m-1})^+\|_{\mathcal{V}_2(\mathcal{O}_t)}^{2+\frac{2(p-n-2)}{np}}\left(A_p(f_0^+,g_0;\mathcal{O}_t)+Lk_m\right).
\end{aligned}
$$

Gronwall's inequality yields that

$$
\begin{aligned}
&\|(u-k_m)^+\|_{\mathcal{V}_2(\mathcal{O}_t)}^2+\|v^{k_m}\|_{0,2;\mathcal{O}_t}^2\\
\le&C_2(\lambda,\kappa,\beta,L,\theta,\varrho,T,n,\epsilon,\epsilon_1)\left(\frac{2^m}{k}\right)^{1+\frac{2(p-n-2)}{np}} &(2.21)\\
&\times\|(u-k_{m-1})^+\|_{\mathcal{V}_2(\mathcal{O}_t)}^{2+\frac{2(p-n-2)}{np}}\left(A_p(f_0^+,g_0;\mathcal{O}_t)+Lk_m\right). &(2.22)
\end{aligned}
$$

Letting $k\ge\bar{k}+\frac{A_p(f_0^+,g_0;\mathcal{O}_t)}{L}$, it follows from (2.21) that

$$
\begin{aligned}
&\|(u-k_m)^+\|_{\mathcal{V}_2(\mathcal{O}_t)}^2+\|v^{k_m}\|_{0,2;\mathcal{O}_t}^2\\
\le&C_3(\lambda,\kappa,\beta,L,\theta,\varrho,T,n,\epsilon,\epsilon_1)\frac{2^{1+\frac{2(p-n-2)}{np}}}{k^{\frac{2(p-n-2)}{np}}}\left(2^{1+\frac{2(p-n-2)}{np}}\right)^{m-1} &(2.23)\\
&\times\|(u-k_{m-1})^+\|_{\mathcal{V}_2(\mathcal{O}_t)}^{2+\frac{2(p-n-2)}{np}}. &(2.24)
\end{aligned}
$$

In terms of $a_m:=\|(u-k_m)^+\|_{\mathcal{V}_2(\mathcal{O}_t)}^2$, $C_0:=C_3(\lambda,\kappa,\beta,L,\theta,\varrho,T,\epsilon,\epsilon_1)\frac{2^{1+\frac{2(p-n-2)}{np}}}{k^{\frac{2(p-n-2)}{np}}}>$

0, $b := 2^{1+\frac{2(p-n-2)}{np}} > 1$ and $\delta := \frac{(p-n-2)}{np} > 0$, we get that

$$a_m \leq C_0 b^{m-1} a_{m-1}^{1+\delta}.$$

Now, let

$$k \geq C_3(\lambda, \kappa, \beta, L, \theta, \varrho, T, \epsilon, \epsilon_1)^{\frac{1}{2\delta}} 2^{(1+2\delta)\left(\frac{1}{2\delta^2}+\frac{1}{2\delta}\right)} \|(u-\bar{k})^+\|_{\mathcal{V}_2(\mathcal{O}_t)}.$$

Then $a_0 \leq C_0^{-\frac{1}{\delta}} b^{-\frac{1}{\delta^2}}$. Therefore, Lemma A.1.1 can be applied to get $\lim_{m\to\infty} a_m = 0$. Along with the above estimates for k this implies that

$$\operatorname{esssup}_{(\omega,t,x)\in\Omega\times\mathcal{O}_t}(u-\bar{k})^+ \leq C\left(\bar{k} + A_p(f_0^+, g_0; \mathcal{O}_t) + \|(u-\bar{k})^+\|_{\mathcal{V}_2(\mathcal{O}_t)}\right), \quad (2.25)$$

where C depends on λ, κ, β, L, ϱ, T, p and n. The estimates of terms $\|(u-\bar{k})^+\|_{\mathcal{V}_2(\mathcal{O}_t)}$ and $\operatorname{esssup}_{(\omega,t,x)\in\Omega\times\mathcal{O}_t}\hat{\xi}^+$ are given in the following Lemma 2.3.3(1) and Lemma 2.3.4(1). Finally we arrive at

$$\begin{aligned}
&\operatorname{esssup}_{(\omega,t,x)\in\Omega\times\mathcal{O}_t} u^+ \\
\leq &C\Big(\operatorname{esssup}_{(\omega,t,x)\in\Omega\times\partial_p\mathcal{O}_t} u^+ + \operatorname{esssup}_{(\omega,t,x)\in\Omega\times\partial_p\mathcal{O}_t}\hat{\xi}^+ \\
&+ A_p(f_0^+, g_0; \mathcal{O}_t) + B_2(f_0^+, g_0; \mathcal{O}_t) + A_p(\hat{f}^+, \hat{g}; \mathcal{O}_t) + B_2(\hat{f}^+, \hat{g}; \mathcal{O}_t)\Big), \quad (2.26)
\end{aligned}$$

where C depends on λ, κ, β, L, ϱ, T, p and n.

(2) For each $t \in [0,T]$, let $k \geq \operatorname{esssup}_{(\omega,t,x)\in\Omega\times\partial_p\mathcal{O}_t} u^+ + \operatorname{esssup}_{(\omega,t,x)\in\Omega\times\mathcal{O}_t}\hat{\xi}^+$. By Theorem A.1.5 we obtain

$$\begin{aligned}
&\|(u-k)^+(t)\|^2 + \int_t^T \|v^k(s)\|^2\, ds \\
&= -2\int_t^T \langle \partial_j (u-k)^+(s), a^{ij}\partial_i(u-k)^+(s) + \sigma^{jr}(s)v^{k,r}s\rangle\, ds \\
&- 2\int_t^T \langle \partial_j(u-k)^+(s), g^{j,k}(s,(u-k)^+(s), \nabla u(s), v^k(s))\rangle\, ds \\
&+ 2\int_t^T \langle (u-k)^+(s), f^k(s,(u-k)^+(s), \nabla u(s), v^k(s))\rangle\, ds \\
&+ 2\int_{\mathcal{O}_t} (u-k)^+(s,x)\,\mu(ds,dx) - 2\int_t^T \langle (u-k)^+(s), v^{r,k}(s)\, dW_s^r\rangle.
\end{aligned}$$

For every $k > l \geq \operatorname{esssup}_{(\omega,t,x)\in\Omega\times\partial_p\mathcal{O}_t} u^+ + \operatorname{esssup}_{(\omega,t,x)\in\Omega\times\mathcal{O}_t}\hat{\xi}^+$, we get

$$\left((u-l)^+ - (u-k)^+\right) 1_{(u>k)} = (k-l)1_{(u>k)},$$

which implies

$$1_{(u>k)} \leq \frac{(u-l)^+}{k-l}.$$

By the assumptions on g and f, and using the same arguments in (2.13), (2.14) and (2.16)-(2.21) we obtain that

$$\|(u-k)^+\|^2_{\mathcal{V}_2(\mathcal{O}_t)} + \|v^k\|^2_{0,2;\mathcal{O}_t} \leq C\frac{A_p(f_0^+, g_0; \mathcal{O}_t)}{(k-l)^{1+\frac{2(p-n-2)}{np}}}\|(u-l)^+\|^{2+\frac{2(p-n-2)}{np}}_{\mathcal{V}_2(\mathcal{O}_t)},$$

where C depends on λ, κ, β, L, ϱ, T and n. By setting $\phi(k) := \|(u-k)^+\|^2_{\mathcal{V}_2(\mathcal{O}_t)}$, $\alpha := 1 + \frac{2(p-n-2)}{np} > 0$, $\zeta := 1 + \frac{p-n-2}{np}$ and $C_1 := CA_p(f_0^+, g_0; \mathcal{O}_t)$, the following statement holds for each $k > l \geq \mathrm{esssup}_{(\omega,t,x)\in\Omega\times\partial_p\mathcal{O}_t} u^+ + \mathrm{esssup}_{(\omega,t,x)\in\Omega\times\mathcal{O}_t}\hat{\xi}^+$:

$$\phi(k) \leq \frac{C_1}{(k-l)^\alpha}\phi(l)^\zeta.$$

If we define $d := C_1^{\frac{1}{\alpha}}\left|\phi(\mathrm{esssup}_{(\omega,t,x)\in\Omega\times\partial_p\mathcal{O}_t} u^+ + \mathrm{esssup}_{(\omega,t,x)\in\Omega\times\mathcal{O}_t}\hat{\xi}^+)\right|^{\frac{\zeta-1}{\alpha}} 2^{\frac{1+\alpha}{\alpha}}$, then by Corollary A.1.2,

$$\|(u-d-\mathrm{esssup}_{(\omega,t,x)\in\Omega\times\partial_p\mathcal{O}_t} u^+ - \mathrm{esssup}_{(\omega,t,x)\in\Omega\times\mathcal{O}_t}\hat{\xi}^+)^+\|_{\mathcal{V}_2(\mathcal{O}_t)} = 0,$$

and so Lemma 2.3.3(2) yields

$$\begin{aligned}\mathrm{esssup}_{(\omega,t,x)\in\Omega\times\mathcal{O}_t} u^+ \leq &\mathrm{esssup}_{(\omega,t,x)\in\Omega\times\partial_p\mathcal{O}_t} u^+ + \mathrm{esssup}_{(\omega,t,x)\in\Omega\times\mathcal{O}_t}\hat{\xi}^+ \\ &+ CA_p(f_0^+, g_0; \mathcal{O}_t)^{\frac{1}{\alpha}} B_2(f_0^+, g_0; \mathcal{O}_t)^{\frac{2(\zeta-1)}{\alpha}},\end{aligned} \tag{2.27}$$

where C depends on λ, κ, β, L, ϱ, n, p and T. Therefore (2.27) and (2.31) yield

$$\begin{aligned}\mathrm{esssup}_{(\omega,t,x)\in\Omega\times\mathcal{O}_t} u^+ \leq &\mathrm{esssup}_{(\omega,t,x)\in\Omega\times\partial_p\mathcal{O}_t} u^+ + \mathrm{esssup}_{(\omega,t,x)\in\Omega\times\partial_p\mathcal{O}_t}\hat{\xi}^+ \\ &+ CA_p(\hat{f}^+, \hat{g}; \mathcal{O}_t)^{\frac{1}{\alpha}} B_2(\hat{f}^+, \hat{g}; \mathcal{O}_t)^{\frac{2(\zeta-1)}{\alpha}} \\ &+ CA_p(f_0^+, g_0; \mathcal{O}_t)^{\frac{1}{\alpha}} B_2(f_0^+, g_0; \mathcal{O}_t)^{\frac{2(\zeta-1)}{\alpha}}.\end{aligned}$$

□

When the domain $\mathcal{O}$ is bounded, $\|\cdot\|_{0,2;Q}$ can be bounded by $\|\cdot\|_{0,p;Q}$ and $\|\cdot\|_{0,\frac{p(n+2)}{p+n+2};Q}$ and we have the following maximum principle for the RBSPDE (2.1) on a bounded domain.

Corollary 2.3.2. *(1) Assume $(\mathcal{A}_1)$-$(\mathcal{A}_5)$ hold and $\mathcal{O}$ is bounded. If the triplet (u, v, μ) is a solution to the RBSPDE (2.1), then*

$$\begin{aligned}&\mathrm{esssup}_{(\omega,t,x)\in\Omega\times Q} u^\pm \\ \leq\, & C\Big(\mathrm{esssup}_{(\omega,t,x)\in\Omega\times\partial_p Q} u^\pm + \mathrm{esssup}_{(\omega,t,x)\in\Omega\times\partial_p Q}\hat{\xi}^\pm \\ &+ A_p(f_0^\pm, g_0; Q) + A_p(\hat{f}^\pm, \hat{g}; Q)\Big),\end{aligned}$$

where the constant C depends only on λ, κ, β, L, ϱ, T, p, n and $|\mathcal{O}|$.

(2) *Suppose that* $(\mathcal{A}_1)$-$(\mathcal{A}_4)$ *and (2.11) hold. Then for each solution* (u, v, μ) *to the RBSPDE (2.1), it holds true that*

$$
\begin{aligned}
&\textit{esssup}_{(\omega,t,x)\in\Omega\times Q} u^{\pm} \\
\leq\, &\textit{esssup}_{(\omega,t,x)\in\Omega\times\partial_p Q} u^{\pm} + \textit{esssup}_{(\omega,t,x)\in\Omega\times\partial_p Q} \hat{\xi}^{\pm} \\
&+ C\left(A_p(f_0^{\pm}, g_0; Q) + A_p(\hat{f}^{\pm}, \hat{g}; Q)\right),
\end{aligned}
$$

where C *depends on* λ, κ, β, L, ϱ, T, p, n, *and* $|\mathcal{O}|$.

Lemma 2.3.3. *(1) Under the same conditions as in Theorem 2.3.1(1), for each* $t \in [0, T]$ *and each* $k \geq \textit{esssup}_{(\omega,t,x)\in\Omega\times\partial_p\mathcal{O}_t} u^+ + \textit{esssup}_{(\omega,t,x)\in\Omega\times\mathcal{O}_t} \hat{\xi}^+$, *we have*

$$
\|(u-k)^+\|_{\mathcal{V}_2(\mathcal{O}_t)} \leq C(B_2(f_0^+, g_0; \mathcal{O}_t) + k),
$$

where C *depends on* λ, κ, β, L, ϱ *and* T.

(2) *Under the same conditions as in Theorem 2.3.1(2), for each* $t \in [0, T]$ *and* $k \geq \textit{esssup}_{(\omega,t,x)\in\Omega\times\partial_p\mathcal{O}_t} u^+ + \textit{esssup}_{(\omega,t,x)\in\Omega\times\mathcal{O}_t} \hat{\xi}^+$ *and we have*

$$
\|(u-k)^+\|_{\mathcal{V}_2(\mathcal{O}_t)} \leq C B_2(f_0^+, g_0; \mathcal{O}_t),
$$

where C *depends on* λ, κ, β, L, ϱ *and* T.

Proof. (1) As in the proof of Theorem 2.3.1, we may assume w.l.o.g that $f(t, x, r, 0, 0)$ is non-increasing in r. For

$$
k \geq \mathrm{esssup}_{(\omega,t,x)\in\Omega\times\partial_p\mathcal{O}_t} u^+ + \mathrm{esssup}_{(\omega,t,x)\in\Omega\times\mathcal{O}_t} \hat{\xi}^+,
$$

we have

$$
\begin{aligned}
&\int_t^T \int_{\mathcal{O}} (u-k)^+ \, \mu(dxds) \\
\leq &\int_t^T \int_{\mathcal{O}} (u-\xi)^+ \, \mu(dxds) + \int_t^T \int_{\mathcal{O}} (\xi - \hat{\xi}^+)^+ \, \mu(dxds) = 0.
\end{aligned}
$$

Applying Theorem A.1.5, we have

$$
\begin{aligned}
&\|(u-k)^+(t)\|^2 + \mathbb{E}\left(\int_t^T \|v^k(s)\|^2 \, ds|\mathcal{F}_t\right) \\
\leq &-2\mathbb{E}\left(\int_t^T \langle \partial_j (u-k)^+(s), a^{ij}\partial_i (u-k)^+(s) + \sigma^{jr}(s) v^{k,r}(s)\rangle \, ds|\mathcal{F}_t\right) \\
&-2\mathbb{E}\left(\int_t^T \langle \partial_j (u-k)^+(s), g^{j,k}\left(s, (u-k)^+(s), \nabla u(s), v^k(s)\right)\rangle \, ds|\mathcal{F}_t\right) \\
&+2\mathbb{E}\left(\int_t^T \langle (u-k)^+(s), f^k\left(s, (u-k)^+(s), \nabla u(s), v^k(s)\right)\rangle \, ds|\mathcal{F}_t\right) \\
:=&K_1 + K_2 + K_3,
\end{aligned}
\tag{2.28}
$$

where $v^{r,k} := v^r 1_{\{u>k\}}$, $g^{j,k}(\cdot,\cdot,\cdot,X,\cdot,\cdot) := g^j(\cdot,\cdot,\cdot,X+k,\cdot,\cdot)$, $f^k(\cdot,\cdot,\cdot,X,\cdot,\cdot) := f(\cdot,\cdot,\cdot,X+k,\cdot,\cdot)$. The quantities K_i $(i=1,2,3)$ can now be estimated by analogy to the constants I_i $(i=1,2,3)$ in the proof of Theorem 2.2.2. Specifically, K_1 can be estimated as I_1, with $u-\hat{\xi}$ and $v-\hat{v}$ being replaced by $(u-k)^+$ and v, respectively; K_2 can be estimated as I_2, without $\hat{g}$ (because we now have no obstacle process involved in), and $u-\hat{\xi}$ and $g^j(s,x,u,\nabla u,v)$ being replaced by $(u-k)^+$ and $g^{j,k}\left(s,x,(u-k)^+(s),\nabla u(s),v^k(s)\right)$, respectively and the estimate for K_3 is similar to that for I_3, without $\hat{f}$ and $u-\hat{\xi}$ and $f(s,x,u,\nabla u,v)$ being replaced by $(u-k)^+$ and $f^k\left(s,x,(u-k)^+,\nabla u,v^k\right)$, respectively. Finally, by $(\mathcal{A}_5)$, $\|g_0^k\|_{0,2;\mathcal{O}_t}$ can be estimated by $\|g_0\|_{0,2;\mathcal{O}_t}+Lk$. This yields the desired result.

(2) The proof is the same as that of (1) if we note that $\|g_0^k\|_{0,2;\mathcal{O}_t}=\|g_0\|_{0,2;\mathcal{O}_t}$ by assumption. □

The following lemma establishes the maximum principle for quasi-linear BSPDE on general domains.

Lemma 2.3.4. *Let (u,v) be a weak solution to the following quasi-linear BSPDE*

$$\begin{cases} -du(t,x) = [\partial_j(a^{ij}\partial_i u(t,x)+\sigma^{jr}v^r(t,x))+f(t,x,u(t,x),\nabla u(t,x),v(t,x)) \\ \qquad\qquad +\nabla\cdot g(t,x,u(t,x),\nabla u(t,x),v(t,x))]\,dt - v^r(t,x)\,dW_t^r, \quad (t,x)\in Q, \\ u(T,x)=G(x), \quad x\in\mathcal{O}. \end{cases} \tag{2.29}$$

(1) If the coefficients satisfy assumptions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, $(\mathcal{A}_3)$ and $(\mathcal{A}_5)$, then for each $t\in[0,T]$ we have

$$\begin{aligned} &\mathrm{esssup}_{(\omega,t,x)\in\Omega\times\mathcal{O}_t} u^\pm \\ &\le C\left(\mathrm{esssup}_{(\omega,t,x)\in\Omega\times\partial_p\mathcal{O}_t}u^\pm + A_p(f_0^\pm,g_0;\mathcal{O}_t)+B_2(f_0^\pm,g_0;\mathcal{O}_t)\right) \end{aligned} \tag{2.30}$$

where C depends on λ, κ, β, L, ϱ, T, p and n;

(2) If $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, $(\mathcal{A}_3)$ and (2.11) hold true, then for each $t\in[0,T]$ we have

$$\begin{aligned} &\mathrm{esssup}_{(\omega,t,x)\in\Omega\times\mathcal{O}_t} u^\pm \\ &\le \mathrm{esssup}_{(\omega,t,x)\in\Omega\times\partial_p\mathcal{O}_t}u^\pm \\ &\quad + CA_p(f_0^\pm,g_0;\mathcal{O}_t)^{\frac{np}{np+2(p-n-2)}}B_2(f_0^\pm,g_0;\mathcal{O}_t)^{\frac{2(p-n-2)}{np+2(p-n-2)}}, \end{aligned} \tag{2.31}$$

where C depends on λ, κ, β, L, ϱ, n, p and T.

Proof. In terms of $\bar{k}=\mathrm{esssup}_{(\omega,t,x)\in\Omega\times\partial_p\mathcal{O}_t}u^+$ the assertion follows by establishing estimates analogous to (2.13)-(2.25) and Lemma 2.3.3(1). □

The proceeding lemmas allow us to establish the comparison principle for the quasi-linear BSPDE on a general domain.

Corollary 2.3.5. *Let (u_i, v_i) be solutions to the quasi-linear BSPDE (2.29) with parameters (f_i, g, G_i, a, σ) respectively, $i = 1, 2$. Suppose that assumptions in Lemma 4.4 hold and that $(u_1 - u_2)^+|_{\partial\mathcal{O}} = 0$. Then if $f_1(t, x, u_2, \nabla u_2, v_2) \le f_2(t, x, u_2, \nabla u_2, v_2)$ $dt \times dx \times d\mathbb{P}$-a.e. and $G_1 \le G_2$ $dx \times d\mathbb{P}$-a.e., we have $u_1 \le u_2$ $dt \times dx \times d\mathbb{P}$-a.e..*

Proof. Let $(\underline{u}, \underline{v}) = (u_1 - u_2, v_1 - v_2)$. Then $(\underline{u}, \underline{v})$ is a solution to the quasi-linear BSPDE (2.29) with parameters $(\underline{f}, \underline{g}, \underline{G}, a, \sigma)$, where

$$\begin{aligned}
&\underline{f}(t, x, \cdot, \cdot, \cdot) = f_1(t, x, \cdot + u_2, \cdot + \nabla u_2, \cdot + v_2) - f_2(t, x, u_2, \nabla u_2, v_2)\\
&\underline{g}(t, x, \cdot, \cdot, \cdot) = g(t, x, \cdot + u_2, \cdot + \nabla u_2, \cdot + v_2) - g(t, x, u_2, \nabla u_2, v_2)\\
&\underline{G} = G_1 - G_2.
\end{aligned}$$

Then we have $\underline{f}_0 := \underline{f}(\cdot, \cdot, 0, 0, 0) \le 0$, $\underline{g}_0 := \underline{g}(\cdot, \cdot, 0, 0, 0) = 0$ and $\operatorname{esssup}_{\Omega \times \partial_p Q} u^+ = 0$. Therefore by Lemma 2.3.3 or Lemma 2.3.4, there holds that $u_1 \le u_2$ $dt \times dx \times d\mathbb{P}$-*a.e.*. ☐

2.3.2. Local Behavior of the Random Field $u^\pm$

The global maximum principle in Theorem 2.3.1 tells us that if the random field $u^\pm$ is bounded on the parabolic boundary, it must be bounded in the whole domain. This section studies the local behavior of $u^\pm$ when it is not necessarily bounded on the parabolic boundary.

Definition 2.3.6. A function ζ is called a cut-off function on the sub-domain $Q' \subset Q$ if it satisfies the following properties:

(1) there exists some smooth function sequence $\{\zeta_m\} \subset C_0^\infty(Q')$ such that ζ_m, $\partial_s \zeta_m$ and $\nabla \zeta_m$ converge to ζ, $\partial_s \zeta$ and $\nabla \zeta$ in $L^\infty(Q')$ respectively;

(2) $\zeta \in [0, 1]$;

(3) there exists a domain $Q'' \subset\subset Q'$ and a nonempty domain $Q''' \subset\subset Q''$ such that
$$\zeta(t, x) = \begin{cases} 0 & \text{if } (t, x) \in Q' \backslash Q'' \\ 1 & \text{if } (t, x) \in Q''', \end{cases}$$
where by $A \subset\subset B$ we mean the closure $\bar{A} \subseteq B$.

We modify the definition of backward stochastic parabolic De Giorgi class in [QT12] as follows.

Definition 2.3.7. We say a function $u \in \mathcal{V}_{2,0}(Q)$ belongs to a backward stochastic parabolic De Giorgi class $BSPDG^\pm(a_0, b_0, k_0, \eta; \delta, Q)$ with

$$(a_0, b_0, k_0, \eta, \delta) \in [0, \infty) \times [0, \infty) \times [0, \infty) \times (n + 2, \infty) \times (0, 1),$$

if for any $Q_{\rho,\tau} := [t_0 - \tau, t_0) \times B_\rho(x_0) \subset Q$ with $(\rho, \tau) \in (0, \delta] \times (0, \delta^2]$, each cut-off function ζ on $Q_{\rho,\tau}$ and for each $k \geq k_0$, we have

$$\|\zeta(u-k)^{\pm}\|^2_{\mathcal{V}_2(Q_{\rho,\tau})} \leq b_0 \Bigg\{ \|(u-k)^{\pm}\|^2_{0,2;Q_{\rho,\tau}} \left(1 + \|\partial_t \zeta\|_{L^\infty(Q_{\rho,\tau})} + \|\nabla \zeta\|^2_{L^\infty(Q_{\rho,\tau})}\right) \\ + \left(k^2 + a_0^2\right) |(u-k)^{\pm} > 0|^{1-\frac{2}{\eta}}_{\infty;Q_{\rho,\tau}} \Bigg\}, \tag{2.32}$$

where

$$\begin{aligned} &|(u-k)^{\pm} > 0|_{\infty;Q_{\rho,\tau}} \\ &:= \operatorname{esssup}_{\omega\in\Omega} \sup_{s\in[t_0-\tau,t_0)} E\left[\int_{[s,t_0)\times B_\rho(x_0)} 1_{\{(u(t,x)-k)^{\pm}>0\}}\, dxdt|\mathcal{F}_s\right]. \end{aligned}$$

Here, we take $(k, \rho, \tau) \in [k_0, \infty) \times (0, \delta] \times (0, \delta^2]$ for given $(k_0, \delta) \in [0, \infty) \times (0, 1)$ in the above definition, instead of $(k, \rho, \tau) \in \mathbb{R} \times (0, 1) \times (0, 1)$ as in [QT12, Definition 5.2]. However, a direct extension of [QT12, Theorem 5.8] yields the following lemma.

Lemma 2.3.8. *Given $k_0^{\pm} \geq 0$, if $u \in BSPDG^{\pm}(a_0^{\pm}, b_0^{\pm}, k_0^{\pm}, \eta; \delta, Q)$, then*

$$\textit{esssup}_{(\omega,t,x)\in\Omega\times Q_{\frac{\rho}{2}}} u^{\pm} \leq 2k_0^{\pm} + C_{\pm}\left\{\rho^{-\frac{n+2}{2}}\|u^{\pm}\|_{0,2;Q_\rho} + a_0^{\pm}\rho^{1-\frac{2+n}{\eta}}\right\},$$

where $Q_\rho := [t_0 - \rho^2, t_0) \times B_\rho(x_0) \subset Q$ with $\rho \in (0, \delta]$ and the constants $C_{\pm}$ depend on $a_0^{\pm}$, $b_0^{\pm}$ and n.

For the solution to the RBSPDE (2.1), we further have the following result.

Lemma 2.3.9. *Let assumptions $(\mathcal{A}_1)$-$(\mathcal{A}_4)$ hold. Suppose (u, v, μ) is a solution to the RBSPDE (1.1). Given $Q_\delta := [t_0 - \delta^2, t_0) \times B_\delta(x_0) \subset Q$ with $\delta \in (0, 1)$, let $k_0^{\pm} = \textit{esssup}_{\Omega\times Q_\delta}\hat{\xi}^{\pm}$. Then we have $u \in BSPDG^{\pm}(a_0^{\pm}, b_0, k_0^{\pm}, \eta; \delta, Q_\delta)$ with $\eta = p$, $a_0^{\pm} = A_p(f_0^{\pm}, g_0; Q_\delta)$ and b_0 depending on λ, κ, β, ϱ, Λ, L, n and p.*

Proof. First we generalize the Itô formula to a local case for the RBSPDE (2.1). For each cut-off function ζ on $Q_{\rho,\tau}$ with $(\rho, \tau) \in (0, \delta] \times (0, \delta^2]$, we can choose a sequence of smooth functions $\{\zeta_m\} \subset C_0^\infty(Q_{\rho,\tau})$ such that ζ_m and its gradients w.r.t. s and x converge uniformly to ζ and its gradient, respectively, as $m \to \infty$.

For $k \geq k_0^+$, Theorem A.1.5 yields that

$$\begin{aligned} &\|(u-k)^+(t)\zeta_m(t)\|^2_{L^2(B_\rho(x_0))} + \int_t^{t_0} \|\zeta_m(s)v^k(s)\|^2_{L^2(B_\rho(x_0))}\, ds \\ =& -2\int_t^{t_0} \langle \zeta_m(s)\partial_s\zeta_m(s), |(u-k)^+(s)|^2\rangle_{B_\rho(x_0)}\, ds \\ &+ 2\int_t^{t_0} \langle \zeta_m^2(s)(u-k)^+(s), f(s, u(s), \nabla u(s), v(s))\rangle_{B_\rho(x_0)}\, ds \\ &- 2\int_t^{t_0} \langle \partial_j(\zeta_m^2(s)(u-k)^+(s)), a^{ij}(s)\partial_i u(s) + \sigma^{jr}(s)v^r(s)\rangle_{B_\rho(x_0)}\, ds \end{aligned}$$

$$
\begin{aligned}
&- 2\int_t^{t_0} \langle \partial_j(\zeta_m^2(s)(u-k)^+(s)), g^j(s,u(s),\nabla u(s),v(s))\rangle_{B_\rho(x_0)}\,ds\\
&- 2\int_t^{t_0} \langle \zeta_m^2(s)(u-k)^+(s), v^{r,k}(s)\rangle_{B_\rho(x_0)}\,dW_s^r\\
&+ 2\int_t^{t_0}\int_{B_\rho(x_0)} (u-k)^+(s,x)\zeta_m^2\,\mu(ds,dx),
\end{aligned}
$$

where $v^{r,k} := v^r 1_{\{u>k\}}$.

Thus by letting $m \to \infty$ and by dominated convergence theorem, we can get

$$
\begin{aligned}
&\|(u-k)^+(t)\zeta(t)\|^2_{L^2(B_\rho(x_0))} + \int_t^{t_0} \|\zeta(s)v^k(s)\|^2_{L^2(B_\rho(x_0))}\,ds\\
&= -2\int_t^{t_0} \langle \zeta(s)\partial_s\zeta(s), |(u-k)^+(s)|^2\rangle_{B_\rho(x_0)}\,ds\\
&+ 2\int_t^{t_0} \langle \zeta^2(s)(u-k)^+(s), f(s,u(s),\nabla u(s),v(s))\rangle_{B_\rho(x_0)}\,ds\\
&- 2\int_t^{t_0} \langle \partial_j(\zeta^2(s)(u-k)^+(s)), a^{ij}(s)\partial_i u(s) + \sigma^{jr}(s)v^r(s)\rangle_{B_\rho(x_0)}\,ds \qquad (2.33)\\
&- 2\int_t^{t_0} \langle \partial_j(\zeta^2(s)(u-k)^+(s)), g^j(s,u(s),\nabla u(s),v(s))\rangle_{B_\rho(x_0)}\,ds\\
&- 2\int_t^{t_0} \langle \zeta^2(s)(u-k)^+(s), v^{r,k}(s)\rangle_{B_\rho(x_0)}\,dW_s^r\\
&+ 2\int_t^{t_0}\int_{B_\rho(x_0)} (u-k)^+(s,x)\zeta^2\,\mu(ds,dx).
\end{aligned}
$$

Taking conditional expectation, we obtain

$$
\begin{aligned}
&\|((u-k)\zeta)^+(t)\|^2_{L^2(B_\rho(x_0))} + \mathbb{E}\left[\int_t^{t_0} \|\zeta(s)v^k(s)\|^2_{L^2(B_\rho(x_0))}\,ds|\mathcal{F}_t\right]\\
&= -2\mathbb{E}\left[\int_t^{t_0} \langle \zeta(s)\partial_s\zeta(s), |(u-k)^+(s)|^2\rangle_{B_\rho(x_0)}\,ds|\mathcal{F}_t\right]\\
&\quad + 2\mathbb{E}\left[\int_t^{t_0} \langle \zeta^2(s)(u-k)^+(s), f^k(s,(u(s)-k)^+,\nabla(u(s)-k)^+,v(s))\rangle_{B_\rho(x_0)}\,ds|\mathcal{F}_t\right]\\
&\quad - 2\mathbb{E}\left[\int_t^{t_0} \langle \partial_j(\zeta^2(s)(u-k)^+(s)), a^{ji}(s)\partial_i u(s) + \sigma^{jr}(s)v^r(s)\right.\\
&\qquad\qquad \left. + g^{j,k}(s,(u(s)-k)^+,\nabla(u(s)-k)^+,v(s))\rangle_{B_\rho(x_0)}\,ds|\mathcal{F}_t\right]\\
&\quad + 2\mathbb{E}\left[\int_t^{t_0}\int_{B_\rho(x_0)} (u-k)^+(s,x)\zeta^2\,\mu(ds,dx)|\mathcal{F}_t\right], \qquad (2.34)
\end{aligned}
$$

where $f^k(\cdot,\cdot,\cdot,X,Y,Z) := f(\cdot,\cdot,\cdot,X+k,Y,Z)$ and $g^{j,k}(\cdot,\cdot,\cdot,X,Y,Z) := g^j(\cdot,\cdot,\cdot,X+k,Y,Z)$. As $k \ge k_0^+$, the last term on the right hand side of (2.34) vanishes. Hence, starting from (2.34), we derive the desired result in a similar way to [QT12, Proposition 5.6]. $\square$

Given $Q_{2\rho} := [t_0 - 4\rho^2, t_0) \times B_{2\rho}(x_0) \subset Q$ with $\rho \in (0,1)$, let $k_0^{\pm} = \text{esssup}_{\Omega \times Q_\rho} \hat{\xi}^{\pm}$. Lemma 2.3.9 shows that $u \in BSPDG^{\pm}(a_0^{\pm}, b_0, k_0^{\pm}, \eta; \rho, Q_\rho)$ with $\eta = p$, $a_0^{\pm} = A_p(f_0^{\pm}, g_0; Q_\rho)$ and b_0 given therein. On the other hand, in view of the local boundedness of weak solutions for BSPDEs ([QT12, Proposition 5.6 and Theorem 5.8]), we have

$$k_0^{\pm} \leq C \left\{ \rho^{-\frac{n+2}{2}} \|\hat{\xi}^{\pm}\|_{0,2;Q_{2\rho}} + A_p(\hat{f}^{\pm}, \hat{g}; Q_{2\rho}) \rho^{1-\frac{2+n}{p}} \right\}$$

with C depending on λ, κ, ϱ, Λ, n and p. Hence, further by Lemmas 2.3.8 and 2.3.9, we obtain finally the local behavior of weak solutions to the RBSPDE (1.1).

Theorem 2.3.10. *Let assumptions $(\mathcal{A}_1)$-$(\mathcal{A}_4)$ hold. Let (u, v, μ) be a weak solution to the RBSPDE (1.1). Given $Q_{2\rho} := [t_0 - 4\rho^2, t_0) \times B_{2\rho}(x_0) \subset Q$ with $\rho \in (0,1)$, we have*

$$\begin{aligned} \text{esssup}_{(\omega,s,x)\in\Omega\times Q_{\frac{\rho}{2}}} u^{\pm} \leq C \Big\{ & \rho^{-\frac{n+2}{2}} (\|u^{\pm}\|_{0,2;Q_\rho} + \|\hat{\xi}^{\pm}\|_{0,2;Q_{2\rho}}) \\ & + \Big(A_p(f_0^{\pm}, g_0; Q_\rho) + A_p(\hat{f}^{\pm}, \hat{g}; Q_{2\rho}) \Big) \rho^{1-\frac{2+n}{p}} \Big\}, \end{aligned}$$

where C is a positive constant depending on λ, κ, β, ϱ, Λ, L, n and p.

3. PART II-1: Mean Field Games with Singular Controls

In this chapter we address the following MFG with singular control

$$\begin{cases} 1. & \text{fix a deterministic function } t \in [0,T] \mapsto \mu_t \in \mathcal{P}(\mathbb{R}^d); \\ 2. & \text{solve the corresponding stochastic singular control problem :} \\ & \inf_{u,Z} \mathbb{E}\left[\int_0^T f(t,X_t,\mu_t,u_t)\,dt + g(X_T,\mu_T) + \int_0^T h(t)\,dZ_t\right], \\ & \text{subject to} \\ & dX_t = b(t,X_t,\mu_t,u_t)\,dt + \sigma(t,X_t,\mu_t,u_t)\,dW_t + c(t)\,dZ_t, \\ 3. & \text{solve } Law(X) = \mu, \text{ where } X \text{ is the optimal state process from 2.,} \end{cases} \tag{3.1}$$

where $u = (u_t)_{t\in[0,T]}$ is the *regular control*, and $Z = (Z_t)_{t\in[0,t]}$ is the *singular control*.

The rest of this chapter is organized as follows: in Section 3.1, we recall the notion of relaxed controls for singular stochastic control problems, introduce MFGs with singular controls and state our main existence of solutions result. The proof is given in Section 3.2. In Section 3.3, we state and prove two approximation results for MFGs with singular controls by MFGs with regular controls. Appendix A.3 recalls known results and definitions that are used throughout this chapter. Appendix A.4 reviews key properties of the M_1 topology.

3.1. Assumptions and the main results

In this section we introduce MFGs with singular controls and state our main existence of solutions result. For a metric space (E,ϱ) we denote by $\mathcal{P}_p(E)$ the class of all probability measures on E with finite moment of p-th order. For $p=0$ we write $\mathcal{P}(E)$ instead of $\mathcal{P}_0(E)$. The set $\mathcal{P}_p(E)$ is endowed with the Wasserstein distance $\mathcal{W}_{p,(E,\varrho)}$; see Definition A.3.1. For a given interval $\mathbb{I}$ we denote by $\mathcal{D}(\mathbb{I})$ the Skorokhod space of all $\mathbb{R}^d$-valued càdlàg functions on $\mathbb{I}$, by $\mathcal{A}(\mathbb{I}) \subset \mathcal{D}(\mathbb{I})$ the subset of nondecreasing functions, by $\mathcal{C}(\mathbb{I}) \subset \mathcal{D}(\mathbb{I})$ the subset of continuous functions, and by $\mathcal{U}(\mathbb{I})$ the set of all measures on $\mathbb{I}\times U$ for some metric space U, whose first marginal is the Lebesgue measure on $\mathbb{I}$, and whose second marginal belongs to $\mathcal{P}(U)$. For reasons that will become clear later we identify processes on $[0,T]$ with processes on the whole real line. For instance, we identify the space $\mathcal{D}(0,T)$ with the space

$$\widetilde{\mathcal{D}}_{0,T}(\mathbb{R}) = \{x \in \mathcal{D}(\mathbb{R}) : x_t = 0 \text{ if } t < 0 \text{ and } x_t = x_T \text{ if } t > T\}.$$

Likewise, we denote by $\widetilde{\mathcal{A}}_{0,T}(\mathbb{R})$ and $\widetilde{\mathcal{C}}_{0,T}(\mathbb{R})$ the subspace of $\widetilde{\mathcal{D}}_{0,T}(\mathbb{R})$ with nondecreasing and continuous paths, respectively. Moreover, we denote by $\widetilde{\mathcal{U}}_{0,T}(\mathbb{R})$

all measures $q(dt, du)$ on $\mathbb{R} \times U$ whose restriction to $[0, T]$ belongs to $\mathcal{U}(0, T)$, and whose restrictions to $(-\infty, 0)$ and (T, ∞) are of the form $q(dt, du) = \delta_{\widetilde{u_0}}(du)dt$ and $q(dt, du) = \delta_{\widetilde{u_T}}(du)dt$ for fixed $\widetilde{u_0} \in U$ and $\widetilde{u_T} \in U$, respectively:

$$
\begin{aligned}
&\widetilde{\mathcal{U}}_{0,T}(\mathbb{R}) \\
&= \left\{ q(dt, du) : q|_{[0,T]\times U} \in \mathcal{U}(0, T), q|_{(-\infty,0)\times U} = \delta_{\widetilde{u_0}}(du)dt, q|_{(T,\infty)\times U} = \delta_{\widetilde{u_T}}(du)dt \right\}.
\end{aligned}
$$

We occasionally drop the subscripts 0 and T if there is no risk of confusion. Throughout this chapter, $C > 0$ denotes a generic constant that may vary from line to line.

3.1.1. Singular stochastic control problems

Before introducing MFGs with singular controls, we informally review stochastic singular control problems of the form:

$$
\begin{cases}
\inf_{u,Z} \mathbb{E}\left[\int_0^T f(t, X_t, u_t)\, dt + g(X_T) + \int_0^T h(t)\, dZ_t\right], \\
\text{subject to} \\
dX_t = b(t, X_t, u_t)\, dt + \sigma(t, X_t, u_t)\, dW_t + c(t)\, dZ_t, \\
X_{0-} = 0.
\end{cases}
\tag{3.2}
$$

where all parameters are measurable in their respective arguments and are such that the control problem makes sense; see, e.g. [HS95] for details.[1] The *regular control* $u = (u_t)_{t\in[0,T]}$ takes values in a compact metric space U, and the *singular control* $Z = (Z_t)_{t\in[0,T]}$ takes values in $\mathbb{R}^d$. For convenience we sometimes write $Z \in \widetilde{\mathcal{A}}(\mathbb{R})$ by which we mean that the sample paths of the stochastic process Z belong to $\widetilde{\mathcal{A}}(\mathbb{R})$. Similarly, we occasionally write $X \in \widetilde{\mathcal{D}}(\mathbb{R})$ and $Y \in \widetilde{\mathcal{C}}(\mathbb{R})$.

Relaxed controls

The existence of optimal *relaxed controls* to stochastic singular control problems has been addressed in [HS95] using the so-called compactification method. We use a similar approach to solve MFGs with singular controls, albeit in different topological setting. The following notion of relaxed controls follows [HS95] where we adopt our convention that all processes are extended to the whole real line.

Definition 3.1.1. The tuple $r = (\Omega, \mathcal{F}, \{\mathcal{F}_t, t \in \mathbb{R}\}, \mathbb{P}, X, \underline{Q}, Z)$ is called a relaxed control if

1. $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \in \mathbb{R}\}, \mathbb{P})$ is a filtered probability space;
2. $\mathbb{P}(X_t = 0, Z_t = 0, \underline{Q}_t(du) = \delta_{\widetilde{u_0}}(du)$ if $t < 0; X_t = X_T, Z_t = Z_T, \underline{Q}_t(du) = \delta_{\widetilde{u_T}}(du)$ if $t > T) = 1$, for some $\widetilde{u_0}, \widetilde{u_T} \in U$;

[1] Our specific assumptions on the model parameters are introduced in Section 3.1.2 below.

3. $\underline{Q} : \mathbb{R} \times \Omega \to \mathcal{P}(U)$ is $\{\mathcal{F}_t, t \in \mathbb{R}\}$ progressively measurable, Z is $\{\mathcal{F}_t, t \in \mathbb{R}\}$ progressively measurable and $Z \in \widetilde{\mathcal{A}}(\mathbb{R})$;

4. X is a $\{\mathcal{F}_t, t \in \mathbb{R}\}$ adapted stochastic process, $X \in \widetilde{\mathcal{D}}(\mathbb{R})$ and for each $\phi \in \mathcal{C}_b^2(\mathbb{R}^d)$, the space of all continuous and bounded functions with continuous and bounded first- and second-order derivatives, $\mathcal{M}^\phi$ is a well defined $\mathbb{P}$ continuous martingale, where

$$\begin{aligned}\mathcal{M}_t^\phi := \phi(X_t) - \int_0^t \int_U \mathcal{L}\phi(s, X_s, u)\, \underline{Q}_s(du)ds - \int_0^t (\partial_x \phi(X_{s-}))^\top c(s) dZ_s \\ - \sum_{0 \le s \le t} \big(\phi(X_s) - \phi(X_{s-}) - (\partial_x \phi(X_{s-}))^\top \triangle X_s\big), \qquad t \in [0, T]\end{aligned}$$

with $\mathcal{L}\phi(t,x,u) := \frac{1}{2} \sum_{ij} a_{ij}(t,x,u) \frac{\partial^2 \phi(x)}{\partial_{x_i} \partial_{x_j}} + \sum_i b_i(t,x,u) \partial_{x_i} \phi(x)$ and $a(t,x,u) = \sigma\sigma^\top(t,x,u)$.

The cost functional corresponding to a relaxed control r is defined by

$$\widetilde{J}(r) = \mathbb{E}^{\mathbb{P}} \left[\int_0^T \int_U f(t, X_t, u)\, \underline{Q}_t(du)dt + \int_0^T h(t)\, dZ_t + g(X_T) \right]. \tag{3.3}$$

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \in \mathbb{R}\}, \mathbb{P}, X, \underline{Q}, Z)$ be a relaxed control. If the process $\underline{Q}$ is of the form $\underline{Q}_t(du) = \delta_{u_t}(du)$, for some progressively measurable U-valued process u, then we call $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \in \mathbb{R}\}, \mathbb{P}, X, u, Z)$ a *strict control.*[2] In particular, any strict control corresponds to a relaxed control. Relaxed control can thus be viewed as a form of mixed strategies over strict controls. In particular, both the cost function and the state dynamics (more precisely, the martingale problem) are linear in relaxed controls. Furthermore, compactness w.r.t. relaxed controls is much easier to verify than compactness w.r.t. strict controls. Under suitable convexity conditions on the model data, the optimization problem over the set of relaxed controls is equivalent to the one over strict controls as shown by the following remark.

Remark 3.1.2. 1. For $(t,x) \in [0,T] \times \mathbb{R}^d$, let

$$K(t,x) = \{(a(t,x,u), b(t,x,u), e) : \ e \ge f(t,x,u), \ u \in U\}.$$

If $K(t,x)$ is convex for each $(t,x) \in [0,T] \times \mathbb{R}^d$, then it can be shown that for each relaxed control, there exists a strict control and a singular control with smaller or equal cost. Indeed, by the proof of [HL90, Theoerem 3.6], for any relaxed control $r = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}, X, \underline{Q}, Z)$, there exists a progressively measurable U-valued process $\bar{u}$ and a $\mathbb{R}^+$-valued process $\bar{v}$ such that for almost all $(t, \omega) \in [0,T] \times \Omega$,

$$\begin{aligned}&\left(\int_U a(t, X_t(\omega), u)\, \underline{Q}_t(\omega, du), \int_U b(t, X_t(\omega), u)\, \underline{Q}_t(\omega, du), \int_U f(t, X_t(\omega), u)\, \underline{Q}_t(\omega, du) \right) \\ &= (a(t, X_t(\omega), \bar{u}_t(\omega)), b(t, X_t(\omega), \bar{u}_t(\omega)), f(t, X_t(\omega), \bar{u}_t(\omega)) + \bar{v}_t(\omega)).\end{aligned} \tag{3.4}$$

[2] If there is no risk of confusion, then we call the processes $\underline{Q}$, respectively u the relaxed, respectively strict control.

Then $\alpha = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}, X, \bar{u}, Z)$ is a strict control with smaller or equal cost.
2. When a and b are linear in u^2 and f is convex in u^2, $K(t,x)$ is convex for each $(t,x) \in [0,T] \times \mathbb{R}^d$.

Canonical state space and disintegration

In what follows, we always assume that Ω is the canonical path space, i.e.

$$\Omega = \widetilde{\mathcal{D}}(\mathbb{R}) \times \widetilde{\mathcal{U}}(\mathbb{R}) \times \widetilde{\mathcal{A}}(\mathbb{R})$$

and that the filtration $\{\mathcal{F}_t, t \in \mathbb{R}\}$ is generated by the coordinate projections X, Q, Z. More precisely, for each $\omega := (x,q,z) \in \Omega$,

$$X(\omega) = x, \qquad Q(\omega) = q, \qquad Z(\omega) = z.$$

and for $t \in [0,T]$, $\mathcal{F}_t := \mathcal{F}_t^X \times \mathcal{F}_t^Q \times \mathcal{F}_t^Z$, where

$$\mathcal{F}_t^X = \sigma(X_s, s \leq t), \qquad \mathcal{F}_t^Q = \sigma(Q(S), S \in \mathcal{B}([0,t] \times U)), \qquad \mathcal{F}_t^Z = \sigma(Z_s, s \leq t);$$

if $t < 0$, then $\mathcal{F}_t := \{\Omega, \emptyset\}$ and if $t > T$, then $\mathcal{F}_t := \mathcal{F}_T$.

The following argument shows that relaxed controls can be defined in terms of projection mappings. In fact, since $[0,T]$ and U are compact, by the definition of $\widetilde{\mathcal{U}}(\mathbb{R})$, each $q \in \widetilde{\mathcal{U}}(\mathbb{R})$ allows for the disintegration

$$q(dt, du) = q_t(du)dt$$

for some measurable $\mathcal{P}(U)$-valued function q_t. By the definition of the space $\widetilde{\mathcal{U}}(\mathbb{R})$ and [Lac15, Lemma 3.2], there exists a $\mathcal{F}_t^Q$-predictable $\mathcal{P}(U)$-valued process Π such that for each $q \in \widetilde{\mathcal{U}}(\mathbb{R})$,

$$\Pi_t(q) = q_t, \ \textit{a.e.}\ t \in [0,T]; \qquad \Pi_t(q) \equiv \delta_{\widetilde{u_0}}, \ t < 0; \qquad \Pi_t(q) \equiv \delta_{\widetilde{u_T}}, \ t > T,$$

where $\widetilde{u_0} \in U$ and $\widetilde{u_T} \in U$ are part of the definition of $\widetilde{\mathcal{U}}(\mathbb{R})$. Hence, the process $Q_t^o := \Pi_t \circ Q$ is $\mathcal{F}_t$-predictable. As a result, for each $\omega = (x,q,z)$,

$$Q(\omega)(dt,du) = q(dt,du) = q_t(du)dt = \Pi_t(q)(du)dt = \Pi_t \circ Q(\omega)(du)dt = Q_t^o(\omega)(du)dt.$$

This yields an adapted disintegration of Q in terms of the $\{\mathcal{F}_t, t \in \mathbb{R}\}$ progressively measurable process

$$Q^o : \mathbb{R} \times \Omega \to \mathcal{P}(U).$$

and hence allows us to define control rules. We notice that it is not appropriate to replace $\widetilde{\mathcal{U}}(\mathbb{R})$ in the definition of the canonical path space by the space of càdlàg $\mathcal{P}(U)$-valued functions as the definition of relaxed controls does not assume any path properties of $t \mapsto \underline{Q}_t$.

Definition 3.1.3. For the canonical path space Ω, the canonical filtration $\{\mathcal{F}_t, t \in \mathbb{R}\}$ and the coordinate projections (X, Q, Z) introduced above, if $r = (\Omega, \mathcal{F}, \{\mathcal{F}_t, t \in \mathbb{R}\}, \mathbb{P}, X, Q^o, Z)$ is a relaxed control in the sense of Definition 3.1.1, then the probability measure $\mathbb{P}$ is called a *control rule*. The associated cost functional is defined as

$$\widehat{J}(\mathbb{P}) := \widetilde{J}(r).$$

Let us denote by $\mathcal{R}$ the class of all the control rules for the stochastic control problem (3.2). Clearly,

$$\inf_{\mathbb{P}\in\mathcal{R}} \widehat{J}(\mathbb{P}) \geq \inf_{\text{relaxed control } r} \widetilde{J}(r).$$

Conversely, for any relaxed control r one can construct a control rule $\mathbb{P} \in \mathcal{R}$ such that $\widehat{J}(\mathbb{P}) = \widetilde{J}(r)$. The proof is standard; it can be found in, e.g. [HS95, Proposition 2.6]. In other words, the optimization problems over relaxed controls and control rules are equivalent. It is hence enough to consider control rules. From now on, we let $(Q_t)_{t\in\mathbb{R}} := (Q^o_t)_{t\in\mathbb{R}}$ for simplicity.

Remark 3.1.4. In [HS95] - with the choice of different topologies and under suitable assumptions on the cost coefficients - it is shown that an optimal control rule exists if $g \equiv 0$. Their method allows for terminal costs only after a modification of the cost function; see [HS95, Remark 2.2 and Section 4] for details. As a byproduct (see Corollary 3.2.9) of our analysis of MFGs, under the same assumptions on the coefficients as in [HS95] we establish the existence of an optimal control rule for terminal cost functions that satisfy a linear growth condition. In Section 3.2.3 we furthermore outline a generalization of the stochastic singular control problem to problems of McKean-Vlasov-type.

3.1.2. Mean field games with singular controls

We are now going to consider MFGs with singular controls of the form (3.1). We again restrict ourselves to relaxed controls. Throughout the chapter, for each $\mu \in \mathcal{P}_p(\widetilde{\mathcal{D}}(\mathbb{R}))$, put $\mu_t = \mu \circ \pi_t^{-1}$, where $\pi_t : x \in \widetilde{\mathcal{D}}(\mathbb{R}) \to x_t$. The first step of solving MFGs is to solve the representative agent's optimal control problem

$$\begin{cases} \inf_{u,Z} \mathbb{E}\left[\int_0^T f(t, X_t, \mu_t, u_t)\,dt + g(X_T, \mu_T) + \int_0^T h(t)\,dZ_t\right] \\ \text{subject to} \\ dX_t = b(t, X_t, \mu_t, u_t)\,dt + \sigma(t, X_t, \mu_t, u_t)\,dW_t + c(t)\,dZ_t, \\ X_{0-} = 0 \end{cases}$$

for any *fixed* mean field measure $\mu \in \mathcal{P}_p(\widetilde{\mathcal{D}}(\mathbb{R}))$. The canonical path space for MFGs with singular controls is

$$\Omega := \widetilde{\mathcal{D}}(\mathbb{R}) \times \widetilde{\mathcal{U}}(\mathbb{R}) \times \widetilde{\mathcal{A}}(\mathbb{R}).$$

We assume that the spaces $\widetilde{\mathcal{D}}(\mathbb{R})$ and $\widetilde{\mathcal{A}}(\mathbb{R})$ are endowed with the M_1 topology. We define a metric on the space $\mathcal{U}(\mathbb{R})$ induced by the Wasserstein distance on compact time intervals by

$$d_{\mathcal{U}(\mathbb{R})}(q^1,q^2) := \mathcal{W}_{p,[0,T]\times U}\left(\frac{q^1}{T},\frac{q^2}{T}\right) \\ +\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\left\{\mathcal{W}_{p,[-(n+1),-n]\times U}(q^1,q^2)+\mathcal{W}_{p,[T+n,T+n+1]\times U}(q^1,q^2)\right\}. \tag{3.5}$$

The space $\widetilde{\mathcal{U}}(\mathbb{R})$ endowed with the metric $d_{\widetilde{\mathcal{U}}(\mathbb{R})} := d_{\mathcal{U}(\mathbb{R})}$ is compact. Furthermore, it is well known [Whi02, Chapter 3] that the spaces $\widetilde{\mathcal{D}}(\mathbb{R})$ and $\widetilde{\mathcal{A}}(\mathbb{R})$ are Polish spaces when endowed with the M_1 topology, and that the σ-algebras on $\widetilde{\mathcal{D}}(\mathbb{R})$ and $\widetilde{\mathcal{A}}(\mathbb{R})$ coincide with the Kolmogorov σ-algebras generated by the coordinate projections.

Definition 3.1.5. A probability measure $\mathbb{P}$ is called a control rule with respect to $\mu \in \mathcal{P}_p(\widetilde{\mathcal{D}}(\mathbb{R}))$ if

1. $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t\in\mathbb{R}\}, \mathbb{P})$ is the canonical probability space and (X, Q, Z) are the coordinate projections;
2. for each $\phi \in \mathcal{C}_b^2(\mathbb{R}^d)$, $\mathcal{M}^{\mu,\phi}$ is a well defined $\mathbb{P}$ continuous martingale, where

$$\mathcal{M}_t^{\mu,\phi} := \phi(X_t) - \int_0^t\int_U \mathcal{L}\phi(s,X_s,\mu_s,u)\,Q_s(du)ds - \int_0^t (\partial_x\phi(X_{s-}))^\top c(s)dZ_s \\ - \sum_{0\le s\le t}\big(\phi(X_s)-\phi(X_{s-})-(\partial_x\phi(X_{s-}))^\top \triangle X_s\big), \qquad t\in[0,T] \tag{3.6}$$

with $\mathcal{L}\phi(t,x,\nu,u) := \frac{1}{2}\sum_{ij}a_{ij}(t,x,\nu,u)\frac{\partial^2\phi(x)}{\partial_{x_i}\partial_{x_j}} + \sum_i b_i(t,x,\nu,u)\partial_{x_i}\phi(x)$ and $a(t,x,\nu,u)=\sigma\sigma^\top(t,x,\nu,u)$, for each $(t,x,\nu,u)\in[0,T]\times\mathbb{R}^d\times\mathcal{P}_p(\mathbb{R}^d)\times U$.

For a fixed measure $\mu \in \mathcal{P}_p(\widetilde{\mathcal{D}}(\mathbb{R}))$, the corresponding set of control rules is denoted by $\mathcal{R}(\mu)$, the cost functional corresponding to a control rule $\mathbb{P}\in\mathcal{R}(\mu)$ is

$$J(\mu,\mathbb{P}) = \mathbb{E}^{\mathbb{P}}\left[\int_0^T\int_U f(t,X_t,\mu_t,u)\,Q_t(du)dt + \int_0^T h(t)\,dZ_t + g(X_T,\mu_T)\right],$$

and the (possibly empty) set of optimal control rules is denoted by

$$\mathcal{R}^*(\mu) := \operatorname{argmin}_{\mathbb{P}\in\mathcal{R}(\mu)} J(\mu,\mathbb{P}).$$

If a probability measure $\mathbb{P}$ satisfies the fixed point property

$$\mathbb{P}\in\mathcal{R}^*(\mathbb{P}\circ X^{-1}),$$

then we call $\mathbb{P} \circ X^{-1}$ or $\mathbb{P}$ or the associated tuple $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}, X, Q, Z)$ a *relaxed solution* to the MFG with singular controls (3.1). Moreover, if $\mathbb{P} \in \mathcal{R}^*(\mathbb{P} \circ X^{-1})$ and $\mathbb{P}(Q(dt, du) = \delta_{\bar{u}_t}(du)dt) = 1$ for some progressively measurable process $\bar{u}$, then we call $\mathbb{P} \circ X^{-1}$ or $\mathbb{P}$ or the associated tuple $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}, X, \bar{u}, Z)$ a *strict solution*.

The following theorem gives sufficient conditions for the existence of a relaxed solution to our MFG. The proof is given in Section 3.2.

Theorem 3.1.6. *For some $\bar{p} > p \geq 1$, we assume that the following conditions are satisfied:*

$\mathcal{A}_1$. *There exists a positive constant C_1 such that $|b| \leq C_1$ and $|a| \leq C_1$; b and σ are measurable in $t \in [0,T]$ and continuous in $(x, \nu, u) \in \mathbb{R}^d \times \mathcal{P}_p(\mathbb{R}^d) \times U$; moreover, b and σ are Lipschitz continuous in $x \in \mathbb{R}^d$, uniformly in $(t, \nu, u) \in [0,T] \times \mathcal{P}_p(\mathbb{R}^d) \times U$.*

$\mathcal{A}_2$. *The functions f and g are measurable in $t \in [0,T]$ and are continuous with respect to $(x, \nu, u) \in \mathbb{R}^d \times \mathcal{P}_p(\mathbb{R}^d) \times U$.*

$\mathcal{A}_3$. *For each $(t, x, \nu, u) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}_p(\mathbb{R}^d) \times U$, there exist strictly positive constants C_2, C_3 and a positive constant C_4 such that*

$$-C_2 \left(1 - |x|^{\bar{p}} + \int_{\mathbb{R}^d} |x|^p \, \nu(dx)\right) \leq g(x, \nu) \leq C_3 \left(1 + |x|^{\bar{p}} + \int_{\mathbb{R}^d} |x|^p \, \nu(dx)\right),$$

and

$$|f(t, x, \nu, u)| \leq C_4 \left(1 + |x|^p + |u|^p + \int_{\mathbb{R}^d} |x|^p \, \nu(dx)\right).$$

$\mathcal{A}_4$. *The functions c and h are continuous and c is strictly positive.*

$\mathcal{A}_5$. *The functions b, σ and f are locally Lipschitz continuous with μ uniformly in (t, x, u), i.e., for $\varphi = b$, σ and f, there exists $C_5 > 0$ such that for each $(t, x, u) \in [0,T] \times \mathbb{R}^d \times U$ and $\nu^1, \nu^2 \in \mathcal{P}_p(\mathbb{R}^d)$ there holds that*

$$|\varphi(t, x, \nu^1, u) - \varphi(t, x, \nu^2, u)| \leq C_5 \Big(1 + L(\mathcal{W}_p(\nu^1, \delta_0), \mathcal{W}_p(\nu^2, \delta_0))\Big) \mathcal{W}_p(\nu^1, \nu^2),$$

where $L(\mathcal{W}_p(\nu^1, \delta_0), \mathcal{W}_p(\nu^2, \delta_0))$ is locally bounded with $\mathcal{W}_p(\nu^1, \delta_0)$ and $\mathcal{W}_p(\nu^2, \delta_0)$.

$\mathcal{A}_6$. *U is a compact metrizable space.*

Under assumptions $\mathcal{A}_1$-$\mathcal{A}_6$, there exists a relaxed solution to the MFGs with singular controls (3.1).

Remark 3.1.7. A typical example where assumption $\mathcal{A}_3$ holds is

$$g(x, \nu) = |x|^{\bar{p}} + \bar{g}(\nu),$$

where $|\bar{g}(\nu)| \leq \int_{\mathbb{R}^d} |y|^p \, \nu(dy)$. This assumption is not needed under a finite fuel constraint on the singular controls. It is needed in order to approximate MFGs with singular controls by MFGs with a finite fuel constraint. The assumption that $c > 0$ is also only needed when passing from finite fuel constrained to unconstrained problems, see Lemma 3.2.12. Assumption $\mathcal{A}_5$ is needed in order to prove the continuity of the cost function and the correspondence $\mathcal{R}$ in μ. A typical example for $\mathcal{A}_5$ is $\int |x|^p \nu(dx)$ or $\int |x|^p \nu(dx) \wedge K$ for some fixed constant K if boundedness is required.

Remark 3.1.8. If we assume for each $(t, x, \nu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_p(\mathbb{R}^d)$, $K(t, x, \nu)$ is convex, where

$$K(t, x, \nu) = \{(a(t, x, \nu, u), b(t, x, \nu, u), e) : \ e \geq f(t, x, \nu, u), \ u \in U\},$$

a strict solution to our MFG can be constructed from a relaxed solution. Let $r^* = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}^*, X, Q, Z)$ is a relaxed solution to MFG. Let $a^*(t, x, u) = a(t, x, \mu_t^*, u)$, $b^*(t, x, u) = b(t, x, \mu_t^*, u)$ and $f^*(t, x, u) = f(t, x, \mu_t^*, u)$, where $\mu^* = \mathbb{P}^* \circ X^{-1}$. Similar to Remark 3.1.2, there exist U-valued process $\bar{u}$ and $\mathbb{R}^+$-valued process $\bar{v}$ such that (3.4) holds with a, b, f replaced by a^*, b^*, f^*, respectively. Define

$$\alpha^* = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q}^*, X, \bar{u}, Z),$$

where $\mathbb{Q}^* = \mathbb{P}^* \circ (X, \delta_{\bar{u}_t}(du)dt, Z)^{-1}$. Then, α^* is a strict solution. The point is that the marginal distribution μ^* does not change when passing from r^* to α^*.

3.2. Proof of the main result

The proof of Theorem 3.1.6 is split into two parts. In Section 3.2.1 we prove the existence of a solution to our MFG under a finite fuel constraint on the singular controls. The general case is established in Section 3.2.2 using an approximation argument.

3.2.1. Existence under a finite fuel constraint

In this section, we prove the existence of a relaxed solution to our MFG under a finite fuel constraint. That is, unless stated otherwise, we restrict the set of admissible singular controls to the set

$$\widetilde{\mathcal{A}}^m(\mathbb{R}) := \{z \in \widetilde{\mathcal{A}}(\mathbb{R}) : z_T \leq m\}, \tag{3.7}$$

for some $m > 0$. By Corollary A.4.5, the set $\widetilde{\mathcal{A}}^m(\mathbb{R})$ is $(\widetilde{\mathcal{D}}(\mathbb{R}), d_{M_1})$ compact.

We start with the following auxiliary result on the tightness of the distributions of the solutions to a certain class of SDEs. The proof uses the definition of the distance $|x - [y, z]|$ of a point x to a line segment $[y, z]$ and the modified strong M_1 oscillation function $\widetilde{w}_s$ introduced in (A.12) and (A.19), respectively.

Proposition 3.2.1. *For each $n \in \mathbb{N}$, on a probability space $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$, let X^n satisfy the following SDE on $[0, T]$:*

$$dX_t^n = b_n(t)\, dt + dM_t^n + dc_n(t), \tag{3.8}$$

where the random coefficients b_n is measurable and bounded uniformly in n, M^n is a continuous martingale with uniformly bounded and absolutely continuous quadratic variation, and c_n is monotone and càdlàg in time a.s. and $\sup_n \mathbb{E}^{\mathbb{P}^n}(|c_n(0)| \vee |c_n(T)|)^{\bar{p}} < \infty$. *Moreover, assume that $X^n_t = 0$ if $t < 0$ and $X^n_t = X^n_T$ if $t > T$. Then, the sequence $\{\mathbb{P}^n \circ (X^n)^{-1}\}_{n\geq 1}$ is relatively compact as a sequence in* $\mathcal{W}_{p,(\widetilde{\mathcal{D}}(\mathbb{R}),d_{M_1})}$.

Proof. By the uniform boundedness of b_n, $\mathbb{E}^{\mathbb{P}^n}(|c_n(0)| \vee |c_n(T)|)^{\bar{p}}$ and the quadratic variation of M^n, there exists a constant C that is independent of n, such that

$$\mathbb{E}^{\mathbb{P}^n} \sup_{0\leq t\leq T} |X^n_t|^{\bar{p}} \leq C < \infty. \tag{3.9}$$

By [Vil09, Definition 6.8(3)] it is thus sufficient to check the tightness of $\{\mathbb{P}^n \circ (X^n)^{-1}\}_{n\geq 1}$. This can be achieved by applying Proposition A.4.7. Indeed, the condition (A.20) holds, due to (3.9). Hence, one only needs to check that for each $\epsilon > 0$ and $\eta > 0$, there exists $\delta > 0$ such that

$$\sup_n \mathbb{P}^n(\widetilde{w}_s(X^n,\delta) \geq \eta) < \epsilon.$$

To this end, we first notice that for each t and $t_1,\ t_2,\ t_3$ satisfying $0 \vee (t-\delta) \leq t_1 < t_2 < t_3 \leq (t+\delta) \wedge T$, the monotonicity of c_n implies

$$\begin{aligned}
&|X^n_{t_2} - [X^n_{t_1}, X^n_{t_3}]| \\
\leq &\left| \int_{t_1}^{t_2} b_n(s)\,ds + M^n_{t_2} - M^n_{t_1} \right| + \left| \int_{t_2}^{t_3} b_n(s)\,ds + M^n_{t_3} - M^n_{t_2} \right| \\
&+ \inf_{0\leq\lambda\leq 1} |c_n(t_2) - \lambda c_n(t_1) - (1-\lambda)c_n(t_3)| \\
= &\left| \int_{t_1}^{t_2} b_n(s)\,ds + M^n_{t_2} - M^n_{t_1} \right| + \left| \int_{t_2}^{t_3} b_n(s)\,ds + M^n_{t_3} - M^n_{t_2} \right|.
\end{aligned}$$

Similarly, for t_1 and t_2 satisfying $0 \leq t_1 < t_2 \leq \delta$,

$$|X^n_{t_1} - [0, X^n_{t_2}]| \leq \left| \int_{t_1}^{t_2} b_n(s)\,ds + M^n_{t_2} - M^n_{t_1} \right|.$$

Therefore,

$$\widetilde{w}_s(X,\delta) \leq\ 3 \sup_t \sup_{t_1,t_2} \left| \int_{t_1}^{t_2} b_n(s)\,ds + M^n_{t_2} - M^n_{t_1} \right|,$$

where the first supremum extends over $0 \leq t \leq T$ and the second one extends over $0 \vee (t-\delta) \leq t_1 \leq t_2 \leq T \wedge (t+\delta)$. By the Markov inequality and the boundedness of b_n and the quadratic variation, this yields

$$\mathbb{P}^n(\widetilde{w}_s(X^n,\delta) \geq \eta) \leq \frac{k(\delta)}{\eta}, \tag{3.10}$$

for some positive function $k(\delta)$ that is independent of n and m with $\lim_{\delta\to 0} k(\delta) = 0$. ☐

The next result shows that the class of all possible control rules is relatively compact. In a subsequent step this will allow us to apply Berge's maximum theorem.

Lemma 3.2.2. *Under assumptions $\mathcal{A}_1$, $\mathcal{A}_4$ and $\mathcal{A}_6$, the set $\bigcup_{\mu\in\mathcal{P}_p(\widetilde{\mathcal{D}}(\mathbb{R}))}\mathcal{R}(\mu)$ is relatively compact in $\mathcal{W}_p$.*

Proof. Let $\{\mu^n\}_{n\geq 1}$ be any sequence in $\mathcal{P}_p(\widetilde{\mathcal{D}}(\mathbb{R}))$ and $\mathbb{P}^n \in \mathcal{R}(\mu^n), n \geq 1$. It is sufficient to show that $\{\mathbb{P}^n \circ X^{-1}\}_{n\geq 1}$, $\{\mathbb{P}^n \circ Q^{-1}\}_{n\geq 1}$ and $\{\mathbb{P}^n \circ Z^{-1}\}_{n\geq 1}$ are relatively compact. Since U and $\widetilde{\mathcal{A}}^m(\mathbb{R})$ are compact by assumption and Corollary A.4.5, respectively, $\{\mathbb{P}^n \circ Q^{-1}\}_{n\geq 1}$ and $\{\mathbb{P}^n \circ Z^{-1}\}_{n\geq 1}$ are tight. Since $\widetilde{\mathcal{U}}(\mathbb{R})$ and $\widetilde{\mathcal{A}}^m(\mathbb{R})$ are compact, these sequences are relatively compact in the topology induced by Wasserstein metric; see [Vil09, Definition 6.8(3)].

It remains to prove the relative compactness of $\{\mathbb{P}^n \circ X^{-1}\}_{n\geq 1}$. Since $\mathbb{P}^n$ is a control rule associated with the measure μ^n, for any n, it follows from Proposition A.3.2 that there exist extensions $(\bar{\Omega}, \bar{\mathcal{F}}, \{\bar{\mathcal{F}}_t, t \in \mathbb{R}\}, \mathbb{Q}^n)$ of the canonical path spaces and processes (X^n, Q^n, Z^n, M^n) defined on it, such that

$$dX^n_t = \int_U b(t, X^n_t, \mu^n_t, u)\, Q^n_t(du)dt + \int_U \sigma(t, X^n_t, \mu^n_t, u)\, M^n(du, dt) + c(t)\, dZ^n_t$$

and

$$\mathbb{P}^n = \mathbb{P}^n \circ (X, Q, Z)^{-1} = \mathbb{Q}^n \circ (X^n, Q^n, Z^n)^{-1},$$

where M^n is a martingale measure on $(\bar{\Omega}, \bar{\mathcal{F}}, \{\bar{\mathcal{F}}_t \in \mathbb{R}\}, \mathbb{Q}^n)$ with intensity Q^n. Relative compactness of $\{\mathbb{P}^n \circ X^{-1}\}_{n\geq 1}$ now reduces to relative compactness of $\{\mathbb{Q}^n \circ (X^n)^{-1}\}_{n\geq 1}$, which is a direct consequence of the preceding Proposition 3.2.1. $\square$

The next result states that the cost functional is continuous on the graph

$$\mathrm{Gr}\mathcal{R} := \{(\mu, \mathbb{P}) \in \mathcal{P}_p(\widetilde{\mathcal{D}}(\mathbb{R})) \times \mathcal{P}_p(\Omega) : \ \mathbb{P} \in \mathcal{R}(\mu)\}.$$

of the multi-function $\mathcal{R}$. This, too, will be needed to apply Berge's maximum theorem below.

Lemma 3.2.3. *Suppose that $\mathcal{A}_1$-$\mathcal{A}_6$ hold. Then $J : Gr\mathcal{R} \to \mathbb{R}$ is continuous.*

Proof. For each $\mu \in \mathcal{P}_p(\widetilde{\mathcal{D}}(\mathbb{R}))$ and $\omega = (x, q, z) \in \Omega$, set

$$\mathcal{J}(\mu, \omega) = \int_0^T \int_U f(t, x_t, \mu_t, u)\, q_t(du)dt + g(x_T, \mu_T) + \int_0^T h(t)\, dz_t. \tag{3.11}$$

Thus

$$J(\mu, \mathbb{P}) = \int_\Omega \mathcal{J}(\mu, \omega)\mathbb{P}(d\omega).$$

In a first step we prove that $\mathcal{J}(\cdot, \cdot)$ is continuous in the first variable; in a second step we prove continuity and a polynomial growth condition in the second variable. The joint continuity of J will be proved in the final step.

Step 1: continuity in μ. Let $\mu^n \to \mu$ in $\mathcal{W}_{p,(\widetilde{\mathcal{D}}(\mathbb{R}),d_{M_1})}$ and recall that $\mu_t^n = \mu^n \circ \pi_t^{-1}$ and $\mu_t = \mu \circ \pi_t^{-1}$, where π is the projection on $\widetilde{\mathcal{D}}(\mathbb{R})$. We consider the first two terms on the r.h.s. in (3.11) separately, starting with the first one. By assumption $\mathcal{A}_5$,

$$
\begin{aligned}
&\left| \int_0^T \int_U f(t, x_t, \mu_t^n, u)\, q_t(du)dt - \int_0^T \int_U f(t, x_t, \mu_t, u)\, q_t(du)dt \right| \\
&\leq C \int_0^T \left(1 + L\left(\mathcal{W}_p(\mu_t^n, \delta_0), \mathcal{W}_p(\mu_t, \delta_0)\right)\right) \mathcal{W}_p(\mu_t^n, \mu_t)\, dt \\
&\leq C \left(\int_0^T \left(1 + L\left(\mathcal{W}_p(\mu_t^n, \delta_0), \mathcal{W}_p(\mu_t, \delta_0)\right)\right)^{\frac{p}{p-1}}\, dt \right)^{1-\frac{1}{p}} \left(\int_0^T \mathcal{W}_p(\mu_t^n, \mu_t)^p\, dt \right)^{\frac{1}{p}}.
\end{aligned}
\tag{3.12}
$$

The convergence $\mu^n \to \mu$ in $\mathcal{W}_{p,(\widetilde{\mathcal{D}}(\mathbb{R}),d_{M_1})}$ implies $\mu^n \to \mu$ weakly. By Skorokhod's representation theorem, there exists $\bar{X}^n$ and $\bar{X}$ defined on some probability space $(\mathbb{Q}, \bar{\Omega}, \bar{\mathcal{F}})$, such that

$$
\mu^n = \mathbb{Q} \circ (\bar{X}^n)^{-1}, \quad \mu = \mathbb{Q} \circ \bar{X}^{-1}
$$

and

$$
d_{M_1}(\bar{X}^n, \bar{X}) \to 0 \quad \mathbb{Q}\text{-}a.s.
$$

Hence, (3.12) implies that

$$
\begin{aligned}
&\left| \int_0^T \int_U f(t, x_t, \mu_t^n, u)\, q_t(du)dt - \int_0^T \int_U f(t, x_t, \mu_t, u)\, q_t(du)dt \right| \\
&\leq C \left(\int_0^T \left(1 + L\left(\mathcal{W}_p(\mathbb{Q} \circ (\bar{X}_t^n)^{-1}, \delta_0), \mathcal{W}_p(\mathbb{Q} \circ \bar{X}_t^{-1}, \delta_0)\right)\right)^{\frac{p}{p-1}}\, dt \right)^{1-\frac{1}{p}} \\
&\quad \times \left(E^{\mathbb{Q}} \int_0^T |\bar{X}_t^n - \bar{X}_t|^p\, dt \right)^{\frac{1}{p}}
\end{aligned}
$$

By Remark A.4.2, we have

$$
\int_0^T |\bar{X}_t^n - \bar{X}_t|^p\, dt \to 0 \qquad \text{a.s. } \mathbb{Q}.
$$

Moreover, we have

$$
\int_0^T |\bar{X}_t^n - \bar{X}_t|^p\, dt \leq 2^p T \left(d_{M_1}(\bar{X}^n, 0)^p + d_{M_1}(\bar{X}, 0)^p \right).
$$

On the other hand,

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}} \left(d_{M_1}(\bar{X}^n, 0)^p + d_{M_1}(\bar{X}, 0)^p \right) &= \int_{\mathcal{D}[0,T]} d_{M_1}(x, 0)^p\, \mu^n(dx) + \int_{\mathcal{D}[0,T]} d_{M_1}(x, 0)^p\, \mu(dx) \\
&\to 2 \int_{\mathcal{D}[0,T]} d_{M_1}(x, 0)^p\, \mu(dx) < \infty.
\end{aligned}
$$

Therefore, dominated convergence yields

$$\mathbb{E}^{\mathbb{Q}} \int_0^T |\bar{X}_t^n - \bar{X}_t|^p \, dt \to 0. \tag{3.13}$$

Since $\sup_n \mathcal{W}_p(\mathbb{Q} \circ (\bar{X}_t^n)^{-1}, \delta_0) < \infty$ it thus follows from the local boundedness of the function L that

$$\left| \int_0^T \int_U f(t, x_t, \mu_t^n, u) \, q_t(du) dt - \int_0^T \int_U f(t, x_t, \mu_t, u) \, q_t(du) dt \right| \to 0, \text{ uniformly in } \omega. \tag{3.14}$$

As for the second term on the r.h.s. in (3.11) recall first that $x^n \to x$ in M_1 implies $x_t^n \to x_t$ for each $t \notin Disc(x)$ and $x_T^n \to x_T$. In particular, the mapping $x \mapsto \varphi(x_T)$ is continuous for any continuous real-valued function φ on $\mathbb{R}^d$. Since any continuous positive function φ on $\mathbb{R}^d$ that satisfies $\varphi(x) \leq C(1 + |x|^p)$, also satisfies

$$\varphi(x_T) \leq C(1 + |x_T|^p) \leq C(1 + d_{M_1}(x, 0)^p)$$

we see that

$$\begin{aligned} &\left| \int_{\mathbb{R}^d} \varphi(x) \, \mu_T^n(dx) - \int_{\mathbb{R}^d} \varphi(x) \, \mu_T(dx) \right| \\ = &\left| \int_{\widetilde{\mathcal{D}}(\mathbb{R})} \varphi(x_T) \, \mu^n(dx) - \int_{\widetilde{\mathcal{D}}(\mathbb{R})} \varphi(x_T) \, \mu(dx) \right| \overset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

More generally, we obtain $\mu_T^n \to \mu_T$ from $\mu^n \to \mu$, which also implies that $g(x_T, \mu_T^n) \to g(x_T, \mu_T)$.

Step 2: continuity in ω. If $\omega^n = (x^n, q^n, z^n) \to \omega = (x, q, z)$, then $x_T^n \to x_T$. In particular,

$$g(x_T^n, \mu_T) \to g(x_T, \mu_T).$$

Moreover, $z^n \to z$ in M_1 implies $z_t^n \to z_t$ for for all continuity points of z and $z_T^n \to z_T$. By the Portmanteau theorem this implies that

$$\int_0^T h(t) \, dz_t^n \to \int_0^T h(t) \, dz_t.$$

Next we show that

$$\int_0^T \int_U f(t, x_t^n, \mu_t, u) \, q_t^n(du) dt \to \int_0^T \int_U f(t, x_t, \mu_t, u) \, q_t(du) dt.$$

By Assumption $\mathcal{A}_2$ the convergence of x^n to x yields $f(t, x_t^n, \mu_t, u) \to f(t, x_t, \mu_t, u)$ for each $t \notin Disc(x)$. From the compactness of U it follows that

$$\sup_{u \in U} |f(t, x_t^n, \mu_t, u) - f(t, x_t, \mu_t, u)| \to 0$$

for each $t \notin Disc(x)$. Since $Disc(x)$ is at most countable this implies

$$\begin{aligned}
&\left| \int_0^T \int_U f(t, x_t^n, \mu_t, u)\, q_t^n(du)dt - \int_0^T \int_U f(t, x_t, \mu_t, u)\, q_t^n(du)dt \right| \\
&\leq \int_0^T \sup_{u \in U} |f(t, x_t^n, \mu_t, u) - f(t, x_t, \mu_t, u)|\, dt \to 0.
\end{aligned}$$

By [Vil09, Definition 6.8], $q^n \to q$ in $d_{\widetilde{\mathcal{U}}(\mathbb{R})}$ implies $q^n \to q$ weakly. Moreover, the first marginal of q^n is Lebesgue measure. Thus, by [JM81, Corollary 2.9], q^n converges to q in the stable topology, which means that $\int \varphi(t,u) q^n(dt, du) \to \int \varphi(t,u) q(dt, du)$ for all bounded and measurable functions φ that are continuous in u. For fixed $(x, \mu) \in \widetilde{\mathcal{D}}(\mathbb{R}) \times \mathcal{P}_p(\widetilde{\mathcal{D}}(\mathbb{R}))$, the compactness of U and the growth condition on f implies the boundedness of f. Hence the definition of stable topology yields that

$$\lim_{n \to \infty} \left| \int_0^T \int_U f(t, x_t, \mu_t, u)\, q_t^n(du)dt - \int_0^T \int_U f(t, x_t, \mu_t, u)\, q_t(du)dt \right| = 0.$$

So we get the convergence

$$\lim_{n \to \infty} \left| \int_0^T \int_U f(t, x_t^n, \mu_t, u)\, q_t^n(du)dt - \int_0^T \int_U f(t, x_t, \mu_t, u)\, q_t(du)dt \right| = 0.$$

Step 3: joint continuity of J. Thus far, we have established the *separate* continuity of the mapping $(\mu, \omega) \to \mathcal{J}(\mu, \omega)$. We are now going to apply [Vil09, Definition 6.8(4)] to prove the *joint* continuity of J.

To this end, notice first that for each fixed $\mu \in \mathcal{P}_p(\widetilde{\mathcal{D}}(\mathbb{R}))$, due to Assumption $\mathcal{A}_3$,

$$\begin{aligned}
&\left| \int_0^T \int_U f(t, x_t, \mu_t, u)\, q_t(du)dt + \int_0^T h(t)\, dz_t \right| \\
&\leq C \left(1 + \int_0^T \int_U \left(1 + |x_t|^p + |u|^p + \int_{\mathbb{R}^d} |y|^p \mu_t(dy) \right) q_t(du)dt + z_T \right) \\
&\leq C \left(1 + d_{M_1}(x, 0)^p + \mathcal{W}_{p,[0,T]\times U} \left(\frac{q}{T}, \delta_0 \right)^p + d_{M_1}(z, 0) + \int_0^T \int_{\mathbb{R}^d} |y|^p\, \mu_t(dy)dt \right) \\
&\leq C \left(1 + d_{M_1}(x, 0)^p + \mathcal{W}_{p,[0,T]\times U} \left(\frac{q}{T}, \delta_0 \right)^p + d_{M_1}(z, 0)^p + \int_{\widetilde{\mathcal{D}}(\mathbb{R})} d_{M_1}(y, 0)^p\, \mu(dy) \right).
\end{aligned}$$

Hence, using the uniform convergence (3.14), it follows from [Vil09, Definition 6.8] that $(\mu^n, \mathbb{P}^n) \to (\mu, \mathbb{P})$ implies that

$$\begin{aligned}
&\Bigg|\mathbb{E}^{\mathbb{P}^n}\left(\int_0^T\int_U f(t,X_t,\mu_t^n,u)\,Q_t(du)dt + \int_0^T h(t)\,dZ_t\right) \\
&-\mathbb{E}^{\mathbb{P}}\left(\int_0^T\int_U f(t,X_t,\mu_t,u)\,Q_t(du)dt + \int_0^T h(t)\,dZ_t\right)\Bigg| \\
\leq &\Bigg|\mathbb{E}^{\mathbb{P}^n}\left(\int_0^T\int_U f(t,X_t,\mu_t^n,u)\,Q_t(du)dt + \int_0^T h(t)\,dZ_t\right) \\
&- \mathbb{E}^{\mathbb{P}^n}\left(\int_0^T\int_U f(t,X_t,\mu_t,u)\,Q_t(du)dt + \int_0^T h(t)\,dZ_t\right)\Bigg| \\
&+\Bigg|\mathbb{E}^{\mathbb{P}^n}\left(\int_0^T\int_U f(t,X_t,\mu_t,u)\,Q_t(du)dt + \int_0^T h(t)\,dZ_t\right) \\
&- \mathbb{E}^{\mathbb{P}}\left(\int_0^T\int_U f(t,X_t,\mu_t,u)\,Q_t(du)dt + \int_0^T h(t)\,dZ_t\right)\Bigg| \\
\to &0.
\end{aligned} \tag{3.15}$$

Since the terminal cost functions is not necessarily Lipschitz continuous we need to argue differently in order to prove the continuous dependence of the expected terminal cost on $(\mu, \mathbb{P})$. First, we notice that for each $\widetilde{p} > \bar{p}$, by the boundedness of b, σ and Z, we have that

$$\sup_n \mathbb{E}^{\mathbb{P}^n} d_{M_1}(X,0)^{\widetilde{p}} \leq C < \infty, \tag{3.16}$$

which implies

$$\lim_{K\to\infty}\sup_n \int_{\{x:d_{M_1}(x,0)>K\}} d_{M_1}(x,0)^{\bar{p}}\,\mathbb{P}^n(dx) = 0. \tag{3.17}$$

By Assumption $\mathcal{A}_3$,

$$|g(x_T,\mu_T)| \leq C\left(1+|x_T|^{\bar{p}} + \int |y|^p \mu_T(dy)\right) \leq C\left(1+|x_T|^{\bar{p}}\right).$$

Together with (3.17) this implies,

$$\mathbb{E}^{\mathbb{P}^n} g(X_T,\mu_T) \to \mathbb{E}^{\mathbb{P}} g(X_T,\mu_T). \tag{3.18}$$

By the tightness of $\{\mathbb{P}^n\}_{n\geq 1}$, for each $\epsilon > 0$, there exists a compact set $K_\epsilon \subseteq \widetilde{\mathcal{D}}(\mathbb{R})$

such that

$$
\begin{aligned}
&\left|\int_{\widetilde{\mathcal{D}}(\mathbb{R})} g(x_T,\mu_T^n)\mathbb{P}^n(dx)-\int_{\widetilde{\mathcal{D}}(\mathbb{R})} g(x_T,\mu_T)\mathbb{P}^n(dx)\right|\\
\le\ &\int_{K_\epsilon}|g(x_T,\mu_T^n)-g(x_T,\mu_T)|\mathbb{P}^n(dx)+\int_{\widetilde{\mathcal{D}}(\mathbb{R})/K_\epsilon}|g(x_T,\mu_T^n)-g(x_T,\mu_T)|\mathbb{P}^n(dx)\\
\le\ &\sup_{x\in K_\epsilon}|g(x_T,\mu_T^n)-g(x_T,\mu_T)|\\
&+\left(\int_{\widetilde{\mathcal{D}}(\mathbb{R})/K_\epsilon}|g(x_T,\mu_T^n)-g(x_T,\mu_T)|^2\mathbb{P}^n(dx)\right)^{\frac12}\left(\sup_n\mathbb{P}^n(\widetilde{\mathcal{D}}(\mathbb{R})/K_\epsilon)\right)^{\frac12}\\
\le\ &\sup_{x\in K_\epsilon}|g(x_T,\mu_T^n)-g(x_T,\mu_T)|+C\epsilon^{\frac12}\quad\text{(by (3.16))}.
\end{aligned}
\tag{3.19}
$$

Thus,

$$
\left|\int_{\widetilde{\mathcal{D}}(\mathbb{R})} g(x_T,\mu_T^n)\mathbb{P}^n(dx)-\int_{\widetilde{\mathcal{D}}(\mathbb{R})} g(x_T,\mu_T)\mathbb{P}^n(dx)\right|\to 0. \tag{3.20}
$$

The convergence (3.15), (3.18) and (3.20) yield the joint continuity of $J(\cdot,\cdot)$. □

Remark 3.2.4. The preceding lemma shows that under a finite fuel constraint the cost functional J is jointly continuous. In general, J is only lower semi-continuous. In fact, for each positive constant K, let $g_K(\cdot) := g(\cdot)\wedge K$ and

$$
\mathcal{J}_K(\mu,\omega) := \int_0^T\int_U f(t,x_t,\mu_t,u)\,q_t(du)dt+g_K(x_T,\mu_T)+\int_0^T h(t)\,dz_t
$$

By assumption $\mathcal{A}_3$, we have

$$
|g_K(x,\mu)|\le 2K+C_2\left(1+\int_{\mathbb{R}^d}|y|^p\mu(dy)\right)\le C\left(1+\int_{\mathbb{R}^d}|y|^p\mu(dy)\right).
$$

So (3.18) and (3.19) still hold with g replaced by g_K while (3.15) still holds for f and h. So $(\mu^n,\mathbb{P}^n)\to(\mu,P)$ implies

$$
\int_\Omega \mathcal{J}_K(\mu^n,\omega)\mathbb{P}^n(d\omega)\to\int_\Omega\mathcal{J}_K(\mu,\omega)\mathbb{P}(d\omega).
$$

Thus, by monotone convergence theorem, we have

$$
\liminf_{n\to\infty}\int_\Omega\mathcal{J}(\mu^n,\omega)\mathbb{P}^n(d\omega)\ge\int_\Omega\mathcal{J}(\mu,\omega)\mathbb{P}(d\omega).
$$

We now recall from [HS95, Proposition 3.1] an equivalent characterization for the set of control rules $\mathcal{R}(\mu)$. This equivalent characterization allows us to verify the martingale property of the state process by verifying the martingale property of its continuous part. Since it is difficult to locate the proof, we give a sketch one in Appendix A.5.

Proposition 3.2.5. *A probability measure $\mathbb{P}$ is a control rule with respect to the given $\mu \in \mathcal{P}_p(\widetilde{\mathcal{D}}(\mathbb{R}))$ if and only if there exists an $\mathcal{F}_t$ adapted process $Y \in \mathcal{C}(0,T)$ on the filtered canonical space $(\Omega, \mathcal{F}, \mathcal{F}_t)$ such that*

(1) $\mathbb{P}(\omega \in \Omega : \ X_t(\omega) = Y_t(\omega) + \int_0^t c(s)\, dZ_s(\omega), t \in [0,T]) = 1$*;*

(2) for each $\phi \in \mathcal{C}_b^2(\mathbb{R}^d)$, $\overline{\mathcal{M}}^{\mu,\phi}$ is a continuous $(\mathbb{P}, \mathcal{F}_t)$ martingale, where

$$\overline{\mathcal{M}}_t^{\mu,\phi} = \phi(Y_t) - \int_0^t \int_U \bar{\mathcal{L}}\phi(s, X_s, Y_s, \mu_s, u)\, Q_s(du)ds, \qquad t \in [0,T] \quad (3.21)$$

with $\bar{\mathcal{L}}\phi(s,x,y,\nu,u) = \sum_i b_i(s,x,\nu,u)\partial_{y_i}\phi(y) + \frac{1}{2}\sum_{ij} a_{ij}(s,x,\nu,u)\frac{\partial^2\phi(y)}{\partial_{y_i}\partial_{y_j}}$ for each $(t,x,y,\nu,u) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}_p(\mathbb{R}^d) \times U$.

The previous characterization of control rules allows us to show that the correspondence $\mathcal{R}$ has a closed graph.

Proposition 3.2.6. *Suppose that $\mathcal{A}_1$ and $\mathcal{A}_4$-$\mathcal{A}_6$ hold. For any sequence $\{\mu^n\}_{n\geq 1} \subseteq \mathcal{P}_p(\widetilde{\mathcal{D}}(\mathbb{R}))$ and $\mu \in \mathcal{P}_p(\widetilde{\mathcal{D}}(\mathbb{R}))$ with $\mu^n \to \mu$ in $\mathcal{W}_{p,(\widetilde{\mathcal{D}}(\mathbb{R}), d_{M_1})}$, if $\mathbb{P}^n \in \mathcal{R}(\mu^n)$ and $\mathbb{P}^n \to \mathbb{P}$ in $\mathcal{W}_p$, then $\mathbb{P} \in \mathcal{R}(\mu)$.*

Proof. In order to verify conditions (1) and (2), notice first that, for each n, there exists a stochastic process $Y^n \in \mathcal{C}(0,T)$ such that

$$\mathbb{P}^n \left(X_t = Y_t^n + \int_0^t c(s)\, dZ_s, t \in [0,T] \right) = 1$$

and such that the corresponding martingale problem is satisfied. In order to show that a similar decomposition and the martingale problem hold under the measure $\mathbb{P}$ we apply Proposition A.3.2. For each n, there exists a probability space $(\Omega^n, \mathcal{F}^n, \mathbb{Q}^n)$ that supports random variables $(\bar{X}^n, \bar{Q}^n, \bar{Z}^n)$ and a martingale measure M^n with intensity $\bar{Q}^n$ such that

$$\mathbb{P}^n = \mathbb{Q}^n \circ (\bar{X}^n, \bar{Q}^n, \bar{Z}^n)^{-1}$$

and

$$d\bar{X}_t^n = \int_U b(t, \bar{X}_t^n, \mu_t^n, u)\, \bar{Q}_s^n(du)ds + \int_U \sigma(t, \bar{X}_t^n, \mu_t^n, u)\, M^n(du, dt) + c(t)d\bar{Z}_t^n.$$

Thus, for each $0 \le s < t \le T$,

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}^n}|Y_t^n - Y_s^n|^4 \\
&= \mathbb{E}^{\mathbb{P}^n}\left|\left(X_t - \int_0^t c(r)\,dZ_r\right) - \left(X_s - \int_0^s c(r)\,dZ_r\right)\right|^4 \\
&= \mathbb{E}^{\mathbb{Q}^n}\left|\left(\bar{X}_t^n - \int_0^t c(r)\,d\bar{Z}_r^n\right) - \left(\bar{X}_s^n - \int_0^s c(r)\,d\bar{Z}_r^n\right)\right|^4 \\
&= \mathbb{E}^{\mathbb{Q}^n}\left|\int_s^t\int_U b(r, \bar{X}_r^n, \mu_r^n, u)\,\bar{Q}_r^n(du)dr + \int_s^t\int_U \sigma(r, \bar{X}_r^n, \mu_r^n, u)\,M^n(du, dr)\right|^4 \\
&\le C|t-s|^2.
\end{aligned}
\tag{3.22}
$$

Hence, Kolmogorov's weak compactness criterion implies the tightness of Y^n. Therefore, taking a subsequence if necessary, the sequence (X, Q, Z, Y^n) of random variables taking values in $\Omega \times \mathcal{C}(0,T)$ has weak limit $(\widehat{X}, \widehat{Q}, \widehat{Z}, \widehat{Y})$ defined on some probability space.

By Skorokhod's representation theorem, there exists a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \mathbb{Q})$ that supports random variables $(\widetilde{X}^n, \widetilde{Q}^n, \widetilde{Z}^n, \widetilde{Y}^n)$ and $(\widetilde{X}, \widetilde{Q}, \widetilde{Z}, \widetilde{Y})$ such that

$$Law(\widetilde{X}^n, \widetilde{Q}^n, \widetilde{Z}^n, \widetilde{Y}^n) = Law(X, Q, Z, Y^n), \quad Law(\widetilde{X}, \widetilde{Q}, \widetilde{Z}, \widetilde{Y}) = Law(\widehat{X}, \widehat{Q}, \widehat{Z}, \widehat{Y})$$

and

$$(\widetilde{X}^n, \widetilde{Q}^n, \widetilde{Z}^n, \widetilde{Y}^n) \to (\widetilde{X}, \widetilde{Q}, \widetilde{Z}, \widetilde{Y}) \quad \mathbb{Q}\text{-a.s.}$$

In particular, $\tilde{Y} \in \mathcal{C}(0,T)$ as the uniform limit of a sequence of continuous processes, and

$$\mathbb{Q}\left(\widetilde{X}_t = \widetilde{Y}_t + \int_0^t c(s)\,d\widetilde{Z}_s, t \in [0,T]\right) = 1.$$

Since $\mathbb{P}^n \to \mathbb{P}$, we have $\mathbb{P} \circ (X, Q, Z)^{-1} = \mathbb{Q} \circ (\widetilde{X}, \widetilde{Q}, \widetilde{Z})^{-1}$. Hence, there exists a stochastic process $Y \in \mathcal{C}(0,T)$ such that

$$\mathbb{P}\left(X_t = Y_t + \int_0^t c(s)\,dZ_s, t \in [0,T]\right) = 1$$

and $\mathbb{P} \circ (X, Q, Z, Y)^{-1} = \mathbb{Q} \circ (\widetilde{X}, \widetilde{Q}, \widetilde{Z}, \widetilde{Y})^{-1}$. Finally, for each $t \in [0,T]$, define

$$\overline{\mathcal{M}}_t^{n,\mu^n,\phi} = \phi(Y_t^n) - \int_0^t\int_U \bar{\mathcal{L}}(s, X_s, Y_s^n, \mu_s^n, u)\,Q_s(du)ds,$$

$$\widetilde{\mathcal{M}}_t^{n,\mu^n,\phi} = \phi(\widetilde{Y}_t^n) - \int_0^t\int_U \bar{\mathcal{L}}(s, \widetilde{X}_s^n, \widetilde{Y}_s^n, \mu_s^n, u)\,\widetilde{Q}_s^n(du)ds,$$

and

$$\widetilde{\mathcal{M}}_t^{\mu,\phi} = \phi(\widetilde{Y}_t) - \int_0^t\int_U \bar{\mathcal{L}}(s, \widetilde{X}_s, \widetilde{Y}_s, \mu_s, u)\,\widetilde{Q}_s(du)ds.$$

For each $0 \leq s < t \leq T$ and each F that is continuous, bounded and $\mathcal{F}_s$-measurable, we have

$$
\begin{aligned}
0 &= \mathbb{E}^{\mathbb{P}^n}\left(\overline{\mathcal{M}}_t^{n,\mu^n,\phi} - \overline{\mathcal{M}}_s^{n,\mu^n,\phi}\right) F(X,Q,Z) \\
&= \mathbb{E}^{\mathbb{Q}}\left(\widetilde{\mathcal{M}}_t^{n,\mu^n,\phi} - \widetilde{\mathcal{M}}_s^{n,\mu^n,\phi}\right) F(\widetilde{X}^n,\widetilde{Q}^n,\widetilde{Z}^n) \qquad (3.23)\\
&\to \mathbb{E}^{\mathbb{Q}}\left(\widetilde{\mathcal{M}}_t^{\mu^*,\phi} - \widetilde{\mathcal{M}}_s^{\mu^*,\phi}\right) F(\widetilde{X},\widetilde{Q},\widetilde{Z}) = \mathbb{E}^{\mathbb{P}}\left(\overline{\mathcal{M}}_t^{\mu,\phi} - \overline{\mathcal{M}}_s^{\mu,\phi}\right) F(X,Q,Z).
\end{aligned}
$$

□

Remark 3.2.7. Note that the proof of Proposition 3.2.6, does not require the finite fuel constraint.

The next corollary shows that the correspondence $\mathcal{R}$ is continuous in the sense of [AB99, Definition 17.2, Theorem 17.20, 17.21].

Corollary 3.2.8. *Suppose that $\mathcal{A}_1$, $\mathcal{A}_4$-$\mathcal{A}_6$ hold. Then, $\mathcal{R} : \mathcal{P}_p(\widetilde{\mathcal{D}}(\mathbb{R})) \to 2^{\mathcal{P}_p(\Omega)}$ is continuous and compact-valued.*

Proof. The lower hemi-continuity of $\mathcal{R}$ can be dealt with as [Lac15, Lemma 4.4] since b and σ are Lipschitz continuous in x. Lemma 3.2.2, Proposition 3.2.6 and [AB99, Theorem 17.20] imply that $\mathcal{R}$ is upper hemi-continuous and compact-valued. □

Corollary 3.2.9. *Under assumptions $\mathcal{A}_1$-$\mathcal{A}_6$, $\mathcal{R}^*(\mu) \neq \emptyset$ for each $\mu \in \mathcal{P}_p(\widetilde{\mathcal{D}}(\mathbb{R}))$ and $\mathcal{R}^*$ is upper hemi-continuous.*

Proof. By [KS91, Section 5.4], for each $\mu \in \mathcal{P}_p(\widetilde{\mathcal{D}}(\mathbb{R}))$ the set $\mathcal{R}(\mu)$ is nonempty. Corollary 3.2.8 implies that $\mathcal{R}$ is compact-valued and continuous. By Lemma 3.2.3, $J : Gr\mathcal{R} \to \mathbb{R}$ is jointly continuous. Thus, [AB99, Theorem 17.31] yields that $\mathcal{R}^*$ is nonempty valued and upper hemi-continuous. □

Remark 3.2.10. Corollary 3.2.9 in fact shows that the stochastic singular control problem (3.2) admits an optimal control rule in the sense of Definition 3.1.3. Using our method, we could have obtained Corollary 3.2.9 under the same assumptions of the coefficients as in [HS95]. We will generalize it to McKean-Vlasov case at the end of this section.

Theorem 3.2.11. *Under assumptions $\mathcal{A}_1$-$\mathcal{A}_6$ and the finite-fuel constraint $Z \in \widetilde{\mathcal{A}}^m(\mathbb{R})$, there exists a relaxed solution to (3.1).*

Proof. From inequality (3.10) in the proof of Proposition 3.2.1, we see that for each $\mu \in \mathcal{P}_p(\widetilde{\mathcal{D}}(\mathbb{R}))$ and $\mathbb{P} \in \mathcal{R}(\mu)$, there exists a nonnegative function $k(\cdot)$ that is independent of μ, such that $\mathbb{P}(\widetilde{w}_s(X,\delta) > \eta) \leq \frac{k(\delta)}{\eta}$ and $\lim_{\delta\to 0} k(\delta) = 0$, where $\widetilde{w}_s$ is the modified oscillation function defined in (A.19).

Let us now define a set-valued map ψ by

$$\psi : \ \mathcal{P}_p(\widetilde{\mathcal{D}}(\mathbb{R})) \to 2^{\mathcal{P}_p(\widetilde{\mathcal{D}}(\mathbb{R}))},$$

$$\mu \mapsto \{\mathbb{P} \circ X^{-1} : \mathbb{P} \in \mathcal{R}^*(\mu)\}, \tag{3.24}$$

and let

$$S = \left\{\mathbb{P} \in \mathcal{P}_p(\widetilde{\mathcal{D}}(\mathbb{R})) : \text{for each } \eta > 0,\ \mathbb{P}(\widetilde{w}_s(X,\delta) > \eta) \leq \frac{k(\delta)}{\eta} \text{ and } \mathbb{E}^{\mathbb{P}} \sup_{0 \leq t \leq T} |X_t|^{\bar{p}} \leq C\right\}$$

where $C < \infty$ denotes the upper bound in (3.9). It can be checked that S is non-empty, relatively compact, convex, and that $\psi(\mu) \subseteq S \subseteq \bar{S}$, for each $\mu \in \widetilde{\mathcal{D}}(\mathbb{R})$. Hence, $\psi : \bar{S} \to 2^{\bar{S}}$. Moreover, by Corollary 3.2.9, ψ is nonempty-valued and upper hemi-continuous. Therefore, [AB99, Corollary 17.55] is applicable by embedding $\mathcal{P}_p(\widetilde{\mathcal{D}}(\mathbb{R}))$ into $\mathcal{M}(\widetilde{\mathcal{D}}(\mathbb{R}))$, the space of all bounded signed measures on $\widetilde{\mathcal{D}}(\mathbb{R})$ endowed with weak convergence topology. □

3.2.2. Existence in the general case

In this section we establish the existence of a solution to MFGs with singular controls for general singular controls $Z \in \widetilde{\mathcal{A}}(\mathbb{R})$. For each m and μ, define

$$\Omega^m = \widetilde{\mathcal{D}}(\mathbb{R}) \times \widetilde{\mathcal{U}}(\mathbb{R}) \times \widetilde{\mathcal{A}}^m(\mathbb{R})$$

and denote by $\mathcal{R}^m(\mu)$ the control rules corresponding to Ω^m and μ, that is, $\mathcal{R}^m(\mu)$ is the subset of probability measures in $\mathcal{R}(\mu)$ that are supported on Ω^m. Denote by $\mathbf{MFG}^m$ the MFGs corresponding to Ω^m. The preceding analysis showed that there exists a solution $\mathbb{P}^{m*}$ to $\mathbf{MFG}^m$, for each m. In what follows,

$$\mu^{m*} := \mathbb{P}^{m*} \circ X^{-1}.$$

The next lemma shows that the sequence $\{\mathbb{P}^{m*}\}_{m \geq 1}$ is relatively compact; the subsequent one shows that any accumulation point is a control rule.

Lemma 3.2.12. *Suppose $\mathcal{A}_1$, $\mathcal{A}_3$, $\mathcal{A}_4$ and $\mathcal{A}_6$ hold. Then there exists a constant $K < \infty$ such that*

$$\sup_m \mathbb{E}^{\mathbb{P}^{m*}} |Z_T|^{\bar{p}} \leq K < \infty.$$

As a consequence, the sequence $\{\mathbb{P}^{m*}\}_{m \geq 1}$ is relatively compact in $\mathcal{W}_{p,\widetilde{\mathcal{D}}(\mathbb{R}) \times \widetilde{\mathcal{U}}(\mathbb{R}) \times \widetilde{\mathcal{A}}(\mathbb{R})}$.

Proof. We recall that $c(\cdot)$ is bounded away from 0. Hence, there exists a constant $C < \infty$ such that, for all $m \in \mathbb{N}$,

$$\mathbb{E}^{\mathbb{P}^{m*}} |Z_T|^{\bar{p}} \leq C\left(1 + \mathbb{E}^{\mathbb{P}^{m*}} |X_T|^{\bar{p}}\right) \tag{3.25}$$

and

$$\mathbb{E}^{\mathbb{P}^{m*}} |X_t|^p \leq C\left(1 + \mathbb{E}^{\mathbb{P}^{m*}} |Z_T|^p\right),\ t \in [0,T]. \tag{3.26}$$

Moreover,

$$
\begin{aligned}
&J(\mu^{m*}, \mathbb{P}^{m*}) \\
&= \mathbb{E}^{\mathbb{P}^{m*}} \left[\int_0^T \int_U f(t, X_t, \mu_t^{m*}, u)\, Q_t(du)dt + g(X_T, \mu_T^{m*}) + \int_0^T h(t)\, dZ_t \right] \\
&\geq \; - C \Bigg(1 + \int_0^T \int_{\mathbb{R}^d} |x|^p \, \mu_t^{m*}(dx)dt + \mathbb{E}^{\mathbb{P}^{m*}} \int_0^T |X_t|^p \, dt \\
&\quad + \mathbb{E}^{\mathbb{P}^{m*}} \int_0^T \int_U |u|^p \, Q_t(du)dt - \mathbb{E}^{\mathbb{P}^{m*}} |X_T|^{\bar{p}} + \int_{\mathbb{R}^d} |x|^p \, \mu_T^{m*}(dx) \\
&\quad + \mathbb{E}^{\mathbb{P}^{m*}} \left| \int_0^T h(t)\, dZ_t \right| \Bigg) \qquad \text{(by assumption } \mathcal{A}_3) \\
&\geq \; - C \Bigg(1 + \int_0^T \int_{\mathbb{R}^d} |x|^p \, \mu_t^{m*}(dx)dt + \mathbb{E}^{\mathbb{P}^{m*}} \int_0^T |X_t|^p \, dt \\
&\quad + \mathbb{E}^{\mathbb{P}^{m*}} \int_0^T \int_U |u|^p \, Q_t(du)dt + \int_{\mathbb{R}^d} |x|^p \, \mu_T^{m*}(dx) \\
&\quad + \mathbb{E}^{\mathbb{P}^{m*}} \left| \int_0^T h(t)\, dZ_t \right| - \mathbb{E}^{\mathbb{P}^{m*}} |Z_T|^{\bar{p}} \Bigg) \qquad \text{(by (3.25))}.
\end{aligned}
$$

Now choose any $\mathbb{P}_0 \in \mathcal{R}^m(\mu^{m*})$ such that $\sup_m J(\mu^{m*}, \mathbb{P}_0) < \infty$ (e.g. $\mathbb{P}_0 \in \mathcal{R}(\mu^{m,*})$ such that $\mathbb{P}_0(Q|_{[0,T]} \equiv \delta_{\widetilde{u}}(du)dt|_{[0,T]}, Z \equiv 0) = 1$ for some $\widetilde{u} \in U$). Then,

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}^{m*}} |Z_T|^{\bar{p}} \\
&\leq J(\mu^{m*}, \mathbb{P}^{m*}) + C \left(1 + \mathbb{E}^{\mathbb{P}^{m*}} \left| \int_0^T h(t)\, dZ_t \right| + \mathbb{E}^{\mathbb{P}^{m*}} \int_0^T |X_t|^p \, dt + \mathbb{E}^{\mathbb{P}^{m*}} |X_T|^p \right) \\
&\leq J(\mu^{m*}, \mathbb{P}_0) + C \left(1 + \mathbb{E}^{\mathbb{P}^{m*}} |Z_T| + \mathbb{E}^{\mathbb{P}^{m*}} |Z_T|^p \right) \\
&\qquad \text{(by (3.26) and the optimality of } \mathbb{P}^{m*}) \\
&\leq C \left(1 + \mathbb{E}^{\mathbb{P}^{m*}} |Z_T| + \mathbb{E}^{\mathbb{P}^{m*}} |Z_T|^p \right).
\end{aligned} \tag{3.27}
$$

Since the measure $\mathbb{P}^{m*}$ is supported on Ω^m, we see that $\mathbb{E}^{\mathbb{P}^{m*}} |Z_T|^{\bar{p}}$ is finite, for each m. In order to see that there exists a uniform upper bound on $\mathbb{E}^{\mathbb{P}^{m*}} |Z_T|^{\bar{p}}$, notice that, independently of m we can choose $M > 0$ large enough such that

$$
\mathbb{E}^{\mathbb{P}^{m*}} |Z_T|^{p_0} \leq M + \frac{1}{4C} \mathbb{E}^{\mathbb{P}^{m*}} |Z_T|^{\bar{p}} \quad (p_0 = 1, p)
$$

Together with (3.27) this yields,

$$
\mathbb{E}^{\mathbb{P}^{m*}} |Z_T|^{\bar{p}} \leq 2C(1 + M) := K.
$$

By [Vil09, Definition 6.8] and Proposition 3.2.1, the relative compactness of $\{\mathbb{P}^{m*}\}_{m \geq 1}$ follows. □

The previous lemma shows that the sequence $\{\mathbb{P}^{m*}\}_{m\geq 1}$ has an accumulation point $\mathbb{P}^*$. Let $\mu^* = \mathbb{P}^* \circ X^{-1}$. Clearly, $\mu^{m*} \to \mu^*$ in $\mathcal{W}_p$ along a subsequence. The following result is an immediate corollary to Proposition 3.2.6 (see Remark 3.2.7).

Lemma 3.2.13. *Suppose that $\mathcal{A}_1$ and $\mathcal{A}_3$-$\mathcal{A}_6$ hold, let $\mathbb{P}^*$ be an accumulation point of the sequence $\{\mathbb{P}^{m*}\}_{m\geq 1}$. Then, $\mathbb{P}^* \in \mathcal{R}(\mu^*)$.*

The next theorem establish the existence of relaxed MFGs solution to (3.1) in the general case, i.e. it proves Theorem 3.1.6.

Theorem 3.2.14. *Suppose $\mathcal{A}_1$-$\mathcal{A}_6$ hold. Then $\mathbb{P}^* \in \mathcal{R}^*(\mu^*)$, i.e., for each $\mathbb{P} \in \mathcal{R}(\mu^*)$ it holds that*

$$J(\mu^*, \mathbb{P}^*) \leq J(\mu^*, \mathbb{P}).$$

Proof. It is sufficient to prove that $J(\mu^*, \mathbb{P}^*) \leq J(\mu^*, \mathbb{P})$ for each $\mathbb{P} \in \mathcal{R}(\mu^*)$ with $J(\mu^*, \mathbb{P}) < \infty$.

By Proposition A.3.2, there exists a filtered probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{F}}_t, \bar{\mathbb{P}})$ on which random variables $(\bar{X}, \bar{Q}, \bar{Z}, M)$ are defined such that $\mathbb{P} = \bar{\mathbb{P}} \circ (\bar{X}, \bar{Q}, \bar{Z})^{-1}$ and

$$d\bar{X}_t = \int_U b(t, \bar{X}_t, \mu_t^*, u)\, \bar{Q}_t(du)dt + \int_U \sigma(t, \bar{X}_t, \mu_t^*, u) M(du, dt) + c(t)\, d\bar{Z}_t, \quad (3.28)$$

where M is a martingale measure with intensity $\bar{Q}$. Using the same argument as in the proof of Lemma 3.2.12 we see that,

$$E^{\mathbb{P}} Z_T^{\bar{p}} = E^{\bar{\mathbb{P}}} \bar{Z}_T^{\bar{p}} < \infty. \quad (3.29)$$

Define $\mathbb{P}^m = \bar{\mathbb{P}} \circ (\bar{X}^m, \bar{Q}, \bar{Z}^m) \in \mathcal{R}^m(\mu^{m*})$, such that $\bar{X}^m$ is the unique strong solution to

$$d\bar{X}_t^m = \int_U b(t, \bar{X}_t^m, \mu_t^{m*}, u)\, \bar{Q}_t(du)dt + \int_U \sigma(t, \bar{X}_t^m, \mu_t^{m*}, u) M(du, dt) + c(t)\, d\bar{Z}_t^m, \quad (3.30)$$

where for each $\bar{\omega} \in \bar{\Omega}$,

$$\bar{Z}_t^m(\bar{\omega}) = \begin{cases} \bar{Z}_t(\bar{\omega}), & \text{if } t < \tau^m(\bar{\omega}) \\ m, & \text{if } t \geq \tau^m(\bar{\omega}), \end{cases}$$

with $\tau^m(\bar{\omega}) = \inf\{t : \bar{Z}_t(\bar{\omega}) > m\}$. Similarly, we can define Z^m. Furthermore, if Z is $\widetilde{\mathcal{A}}^m(\mathbb{R})$ valued, we have $Z = Z^m$. Hence,

$$\begin{aligned} &\mathbb{E}^{\bar{\mathbb{P}}} \sup_{0\leq t\leq T} \left| \int_0^t c(s)\, d\bar{Z}_s - \int_0^t c(s)\, d\bar{Z}_s^m \right| \\ &= \mathbb{E}^{\mathbb{P}} \sup_{0\leq t\leq T} \left| \int_0^t c(s)\, dZ_s - \int_0^t c(s)\, dZ_s^m \right| \\ &= \int_{\widetilde{\mathcal{A}}(\mathbb{R})\setminus\widetilde{\mathcal{A}}^m(\mathbb{R})} \sup_{0\leq t\leq T} \left| \int_0^t c(s)\, dZ_s(\omega) - \int_0^t c(s)\, dZ_s^m(\omega) \right| \mathbb{P}(d\omega). \end{aligned} \quad (3.31)$$

By Hölder's inequality,

$$
\begin{aligned}
&\int_{\widetilde{\mathcal{A}}(\mathbb{R})\setminus\widetilde{\mathcal{A}}^m(\mathbb{R})} \sup_{0\le t\le T}\left|\int_0^t c(s)\,dZ_s(\omega)-\int_0^t c(s)\,dZ_s^m(\omega)\right|\mathbb{P}(d\omega)\\
&\le \left|\int_{\widetilde{\mathcal{A}}(\mathbb{R})\setminus\widetilde{\mathcal{A}}^m(\mathbb{R})}\int_0^T c(t)\,dZ_t(\omega)\mathbb{P}(d\omega)+\int_{\widetilde{\mathcal{A}}(\mathbb{R})\setminus\widetilde{\mathcal{A}}^m(\mathbb{R})}\int_0^T c(t)\,dZ_t^m(\omega)\mathbb{P}(d\omega)\right|\\
&\le C\left(E^{\mathbb{P}}Z_T^p\right)^{\frac{1}{p}}\mathbb{P}(\widetilde{\mathcal{A}}(\mathbb{R})\setminus\widetilde{\mathcal{A}}^m(\mathbb{R}))^{1-\frac{1}{p}}+C\left(E^{\mathbb{P}}(Z_T^m)^p\right)^{\frac{1}{p}}\mathbb{P}(\widetilde{\mathcal{A}}(\mathbb{R})\setminus\widetilde{\mathcal{A}}^m(\mathbb{R}))^{1-\frac{1}{p}}\\
&\le C\left(E^{\mathbb{P}}Z_T^p\right)^{\frac{1}{p}}\mathbb{P}(\widetilde{\mathcal{A}}(\mathbb{R})\setminus\widetilde{\mathcal{A}}^m(\mathbb{R}))^{1-\frac{1}{p}}.
\end{aligned}
$$

Since $\widetilde{\mathcal{A}}^m(\mathbb{R})\uparrow\widetilde{\mathcal{A}}(\mathbb{R})$ implies $\mathbb{P}(\widetilde{\mathcal{A}}(\mathbb{R})\setminus\widetilde{\mathcal{A}}^m(\mathbb{R}))\to 0$ we get,

$$
\mathbb{E}^{\bar{\mathbb{P}}}\sup_{0\le t\le T}\left|\int_0^t c(s)\,d\bar{Z}_s-\int_0^t c(s)\,d\bar{Z}_s^m\right|\to 0. \tag{3.32}
$$

Similarly,

$$
\mathbb{E}^{\bar{\mathbb{P}}}\left|\int_0^T h(t)\,d\bar{Z}_t-\int_0^T h(t)\,d\bar{Z}_t^m\right|\to 0. \tag{3.33}
$$

By (3.28), (3.30) and (3.32), the Lipschitz continuity of b and σ in x and μ and the Burkholder-Davis-Gundy inequality, standard estimate of SDE yields that

$$
\lim_{m\to\infty}\mathbb{E}^{\bar{\mathbb{P}}}\sup_{0\le t\le T}\left|\bar{X}_t^m-\bar{X}_t\right|=0. \tag{3.34}
$$

By (3.33), (3.34), $\mu^{m*}\to\mu^*$ in $\mathcal{W}_{p,(\widetilde{\mathcal{D}}(\mathbb{R}),d_{M_1})}$ and the same arguments as in the proof of Lemma 3.2.3, we get

$$
\begin{aligned}
&\mathbb{E}^{\bar{\mathbb{P}}}\left(\int_0^T f(t,\bar{X}_t^m,\mu_t^{m*},u)\,\bar{Q}_t(du)dt+g(\bar{X}_T^m,\mu_T^{m*})+\int_0^T h(t)\,d\bar{Z}_t^m\right)\\
\to\ &\mathbb{E}^{\bar{\mathbb{P}}}\left(\int_0^T f(t,\bar{X}_t,\mu_t^*,u)\,\bar{Q}_t(du)dt+g(\bar{X}_T,\mu_T^*)+\int_0^T h(t)\,d\bar{Z}_t\right).
\end{aligned}
$$

This shows that

$$
J(\mu^{m*},\mathbb{P}^m)\to J(\mu^*,\mathbb{P}).
$$

Moreover, by Remark 3.2.4, $\liminf_{m\to\infty}J(\mu^{m*},\mathbb{P}^{m*})\ge J(\mu^*,\mathbb{P}^*)$. Hence,

$$
J(\mu^*,\mathbb{P})=\lim_{m\to\infty}J(\mu^{m*},\mathbb{P}^m)\ge\liminf_{m\to\infty}J(\mu^{m*},\mathbb{P}^{m*})\ge J(\mu^*,\mathbb{P}^*).
$$

□

3.2.3. Related McKean-Vlasov stochastic singular control problem

MFGs and control problems of McKean-Vlasov type are compared in [CDL13]. The literatures on McKean-Vlasov singular control focus on necessary conditions for optimality; the existence of an optimal solution is typically assumed. An exception is the recent work [Lac17] that established a similar existence result for regular (relaxed) controls. In this section we outline how our results on MFGs with singular controls can be used to establish the existence of an optimal control to the following McKean-Vlasov stochastic singular control problem:

$$\begin{aligned} &\min_{u,Z} J(u,Z) \\ =& \min_{u,Z} \mathbb{E}\left[\int_0^T f(t,X_t,Law(X_t),u_t)\,dt + g(X_T,Law(X_T)) + \int_0^T h(t)\,dZ_t\right] \end{aligned} \tag{3.35}$$

subject to

$$dX_t = b(t,X_t,Law(X_t),u_t)\,dt + \sigma(t,X_t,Law(X_t),u_t)\,dW_t + c(t)\,dZ_t,\ \ t\in[0,T]. \tag{3.36}$$

To this end, we first introduce relaxed controls and control rules similar to Section 3.1.

Definition 3.2.15. We call $(\Omega,\mathcal{F},\{\mathcal{F}_t,t\in\mathbb{R}\},\mathbb{P},X,\underline{Q},Z)$ a relaxed control to McKean-Vlasov stochastic singular control problem (3.35)-(3.36) if it satisfies items 1, 2 and 3 in Definition 3.1.1 and

4' $\left(\mathcal{M}^{\mathbb{P},\phi},\{\mathcal{F}_t,t\geq 0\},\mathbb{P}\right)$ is a well defined continuous martingale, where

$$\begin{aligned} \mathcal{M}_t^{\mathbb{P},\phi} = \phi(X_t) &- \int_0^t\int_U \phi'(X_s)b(s,X_s,\mathbb{P}\circ X_s^{-1},u)\,\underline{Q}_s(du)ds \\ &- \frac{1}{2}\int_0^t\int_U \phi'(X_s)a(s,X_s,\mathbb{P}\circ X_s^{-1},u)\,\underline{Q}_s(du)ds \\ &- \int_0^t \phi'(X_{s-})c(s)\,dZ_s \\ &- \sum_{0\leq s\leq t}\left(\phi(X_s)-\phi(X_{s-})-\phi'(X_{s-})\Delta X_s\right),\ \ t\in[0,T]. \end{aligned} \tag{3.37}$$

For each relaxed control $r=(\Omega,\mathcal{F},\{\mathcal{F}_t,t\in\mathbb{R}\},\mathbb{P},X,\underline{Q},Z)$, we define the corresponding cost functional by

$$J(r) = \mathbb{E}^{\mathbb{P}}\left[\int_0^T\int_U f\left(t,X_t,\mathbb{P}\circ X_t^{-1},u\right)\,\underline{Q}_t(du)dt + g\left(X_T,\mathbb{P}\circ X_T^{-1}\right) + \int_0^T h(t)\,dZ_t\right]. \tag{3.38}$$

We still denote by $\Omega := \widetilde{\mathcal{D}}(\mathbb{R})\times\widetilde{\mathcal{U}}(\mathbb{R})\times\widetilde{\mathcal{A}}(\mathbb{R})$ the canonical space, $\mathcal{F}_t$ the canonical filtration and (X,Q,Z) the coordinate projections with the associated predictable disintegration Q^o, as introduced in Section 3.1. The notion of control rules can be defined similarly as that in Definition 3.1.3. Denote by $\mathcal{R}$ all the control rules. For $\mathbb{P}\in\mathcal{R}$, the corresponding cost functional is defined as in (3.38).

Using straightforward modifications of arguments given in the proof of [HS95, Proposition 2.6] we see that our optimization problems over relaxed controls and over control rules are equivalent. Once the optimal control rule is established, under the same additional assumption as in Remark 3.1.8, we can establish a strict optimal control from the optimal control rule. The next two theorems prove the existence of an optimal control under a finite-fuel constraint $Z \in \widetilde{\mathcal{A}}^m(\mathbb{R})$ on the singular controls; see (3.7). The existence results can then be extended to the general unconstraint case. We do not give a formal proof as the arguments are exactly the same as in the preceding subsection.

Theorem 3.2.16. *Suppose $\mathcal{A}_4$, $\mathcal{A}_5$ hold and $\mathcal{A}_1$ holds without Lipschitz continuity of b and σ on x. Under the finite-fuel constraint* (3.7)*, the set $\mathcal{R}$ is non-empty.*

Proof. For each $\mu \in \mathcal{P}_p(\widetilde{\mathcal{D}}(\mathbb{R}))$, there exists a solution to the martingale problem $\mathcal{M}^{\mu,\phi}$, where $\mathcal{M}^{\mu,\phi}$ is defined in (3.6). Thus, we define a set-valued map Φ on $\mathcal{P}_p(\widetilde{\mathcal{D}}(\mathbb{R}))$ with non-empty convex images by

$$\Phi : \mu \to \{\mathbb{P} \circ X^{-1} : \mathbb{P} \in \mathcal{R}(\mu)\},$$

where $\mathcal{R}(\mu)$ is the control rule with μ as in the previous section.

The compactness of $\Phi(\mu)$ for each $\mu \in \mathcal{P}_p(\widetilde{\mathcal{D}}(\mathbb{R}))$ and the upper hemi-continuity of Φ are results of the compactness of $\mathcal{R}(\mu)$ for each $\mu \in \mathcal{P}_p(\widetilde{\mathcal{D}}(\mathbb{R}))$ and upper hemi-continuity of $\mathcal{R}(\cdot)$, respectively, which are direct results of Corollary 3.2.8.[3] By analogy to the proof of Theorem 3.2.11 we can define a non-empty, compact, convex set $\bar{S} \subset \mathcal{P}_p(\widetilde{\mathcal{D}}(\mathbb{R}))$ such that $\Phi : \bar{S} \to 2^{\bar{S}}$. Hence, Φ has a fixed point, due to [AB99, Corollary 17.55]. □

Theorem 3.2.17. *Suppose $\mathcal{A}_3$-$\mathcal{A}_6$ hold and that $\mathcal{A}_1$ holds without Lipschitz assumptions on b and σ in x, and that $\mathcal{A}_2$ holds with the continuity of f and g being replaced by lower semi-continuity. Under the finite-fuel constraint* (3.7)*, there exists an optimal control rule, that is, there exists $\mathbb{P}^* \in \mathcal{R}$ such that*

$$J(\mathbb{P}^*) \leq J(\mathbb{P}) \quad \textit{for all} \quad \mathbb{P} \in \mathcal{R}.$$

Proof. It is sufficient to prove $\mathcal{R}$ is compact and J is lower semi-continuous. The former one can be achieved by the same way to Corollary 3.2.8. As for the lower semi-continuity, note that f and g can be approximated by continuous functions f_N and g_N increasingly. For f_N and g_N, by the same way as that in the proof of

[3] Note that we only need upper hemi-continuity of $\mathcal{R}(\cdot)$, so Lipschtiz assumptions on b and σ are not necessary.

Lemma 3.2.3, one has

$$\liminf_{n\to\infty} \mathbb{E}^{\mathbb{P}^n}\left[\int_0^T \int_U f_N(t, X_t, \mathbb{P}^n \circ X_t^{-1}, u)\, Q_t(du)dt \right.$$
$$\left. + g_N(X_T, \mathbb{P}^n \circ X_T^{-1}) + \int_0^T h(t)\, dZ_t\right]$$
$$\to \mathbb{E}^{\mathbb{P}}\left[\int_0^T \int_U f_N(t, X_t, \mathbb{P} \circ X_t^{-1}, u)\, Q_t(du)dt + g_N(X_T, \mathbb{P} \circ X_T^{-1}) + \int_0^T h(t)\, dZ_t\right].$$

Thus, monotone convergence implies the lower semi-continuity of J. $\square$

3.3. MFGs with regular controls and MFGs with singular controls

In this section we establish two approximation results for a class of MFGs with singular controls under finite-fuel constraints. For the reasons outlined in Remark 3.3.2 below we restrict ourselves to MFGs without terminal cost or singular control cost. More precisely, we consider MFGs with singular controls of the form:

$$\begin{cases} 1. \quad \text{fix a deterministic measure } \mu \in \mathcal{P}_p(\widetilde{\mathcal{D}}_{0,T+\epsilon}(\mathbb{R})); \\ 2. \quad \text{solve the corresponding stochastic singular control problem :} \\ \quad\ \inf_{u,Z} \mathbb{E}\left[\int_0^T f(t, X_t, \mu_t, u_t)dt\right] \\ \quad\ \text{subject to} \\ \quad\ dX_t = b(t, X_t, \mu_t, u_t)\, dt + \sigma(t, X_t, \mu_t, u_t)dW_t + c(t)\, dZ_t,\ \ t \in [0, T+\epsilon]; \\ 3. \quad \text{solve } \mu = Law(X), \text{ where } X \text{ is the optimal state process from 2.,} \end{cases} \tag{3.39}$$

for some fixed $\epsilon > 0$ under the finite-fuel constraint $Z \in \widetilde{\mathcal{A}}^m_{0,T}(\mathbb{R})$. The reason we define the state process on the time interval $[0, T+\epsilon]$ is that we approximate the singular controls by absolutely continuous ones that are most naturally regarded as elements of $\widetilde{\mathcal{D}}_{0,T+\epsilon}(\mathbb{R})$ rather than $\widetilde{\mathcal{D}}_{0,T}(\mathbb{R})$.

3.3.1. Solving MFGs with singular controls using MFGs with regular controls

In this section we establish an approximation of (relaxed) solutions results for the MFGs (3.39) under a finite-fuel constraint by (relaxed) solutions to MFGs with only regular controls. To this end, we associate with each singular control $Z \in \widetilde{\mathcal{A}}^m_{0,T}(\mathbb{R})$ the sequence of absolutely continuous controls

$$Z^{[n]}_t = n \int_{(t-\frac{1}{n})}^t Z_s\, ds \quad (t \in \mathbb{R}, n \in \mathbb{N})\,^4. \tag{3.40}$$

[4] This approximation has been widely used in singular control literature. In particular, it has been used recently in the stability of optimal liquidation problem in [BBF17]

Then, $Z^{[n]} \in \widetilde{\mathcal{A}}^m_{0,T+\epsilon}(\mathbb{R})$ for all sufficiently large $n \in \mathbb{N}$. Since each $Z^{[n]}$ is absolutely continuous and Z is càdlàg we cannot expect convergence of Z^n to Z in the Skorokhod J_1 topology in general. However, by Proposition A.4.1 (3.) and the discussion before Proposition A.4.4 we do know that

$$Z^{[n]} \to Z \quad \text{a.s. in} \quad \left(\widetilde{\mathcal{D}}_{0,T+\epsilon}(\mathbb{R}), d_{M_1}\right).$$

For each n, we consider the following finite-fuel constrained MFGs denoted by $\mathbf{MFG}^{[n]}$:

$$\begin{cases} 1. \quad \text{fix a deterministic measure } \mu \in \mathcal{P}_p(\widetilde{\mathcal{D}}_{0,T+\epsilon}(\mathbb{R})); \\ 2. \quad \text{solve the corresponding stochastic control problem :} \\ \quad\ \inf_{u,Z} \mathbb{E}\left[\int_0^T f(t, X_t^{[n]}, \mu_t, u_t)dt\right] \\ \quad\ \text{subject to} \\ \quad\ dX_t^{[n]} = b(t, X_t^{[n]}, \mu_t, u_t)dt + \sigma(t, X_t^{[n]}, \mu_t, u_t)dW_t + c(t)\, dZ_t^{[n]}, \ t \in [0, T+\epsilon] \\ \quad\ X_0^{[n]} = 0 \\ \quad\ Z_t^{[n]} = n\int_{(t-\frac{1}{n})}^t Z_s\, ds; \\ 3. \quad \text{solve } \mu = Law(X^{[n]}), \text{ where } X^{[n]} \text{ is the optimal state process from 2.} \end{cases} \tag{3.41}$$

Definition 3.3.1. We call the vector $r^n = (\Omega, \mathcal{F}, \{\mathcal{F}_t, t \in \mathbb{R}\}, \mathbb{P}, X, \underline{Q}, Z^{[n]})$ a *relaxed control* with respect to μ for some $\mu \in \mathcal{P}_p(\widetilde{\mathcal{D}}_{0,T+\epsilon}(\mathbb{R}))$ if $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \in \mathbb{R}\}, \mathbb{P}, X, \underline{Q}, Z)$ satisfies 1.-3. in Definition 3.1.1 with item 4 being replaced by

4′. X is a $\{\mathcal{F}_t, t \in \mathbb{R}\}$ adapted stochastic process and $X \in \widetilde{\mathcal{D}}_{0,T+\epsilon}(\mathbb{R})$ such that for each $\phi \in \mathcal{C}_b^2(\mathbb{R}^d)$, $\mathcal{M}^{[n],\mu,\phi}$ is a well defined $\mathbb{P}$ continuous martingale, where

$$\begin{aligned} \mathcal{M}_t^{[n],\mu,\phi} := \phi(X_t) &- \int_0^t \int_U \mathcal{L}\phi(s, X_s, \mu_s, u)\, \underline{Q}_s(du)ds \\ &- \int_0^t (\partial_x \phi(X_s))^\top c(s)\, dZ_s^{[n]}, \end{aligned} \tag{3.42}$$

with $\mathcal{L}$ defined as in Definition 3.1.5.

The probability measure $\mathbb{P}$ is called a control rule if $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \in \mathbb{R}\}, \mathbb{P}, X, Q^o, Z^{[n]})$ is a relaxed control with $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \in \mathbb{R}\})$ being the filtered canonical space with

$$\Omega := \widetilde{\mathcal{D}}_{0,T+\epsilon}(\mathbb{R}) \times \widetilde{\mathcal{U}}_{0,T+\epsilon}(\mathbb{R}) \times \widetilde{\mathcal{A}}^m_{0,T}(\mathbb{R})$$

and (X, Q, Z) being the coordinate projections on $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \in \mathbb{R}\})$ and Q^o being the disintegration of Q as in Section 3.1.1.

Remark 3.3.2. If Z is discontinuous at T, then $Z^{[n]}$ may not converge to Z in $\widetilde{\mathcal{D}}_{0,T}(\mathbb{R})$ but only in $\widetilde{\mathcal{D}}_{0,T+\epsilon}(\mathbb{R})$. Likewise, the associated sequence of the state processes may only converge in $\widetilde{\mathcal{D}}_{0,T+\epsilon}(\mathbb{R})$. The possible discontinuity at the terminal time T is also the reason why there is no terminal cost and no cost from singular control in this section. If we assume that T is always a continuous point, then terminal costs and costs from singular controls are permitted. In this case, one may as well allow unbounded singular controls.

For each fixed n and μ, denote by $\mathcal{R}^{[n]}(\mu)$ the set of all the control rules for $\mathbf{MFG}^{[n]}$, and define the cost functional corresponding to the control rule $\mathbb{P} \in \mathcal{R}^{[n]}(\mu)$ by

$$J^{[n]}(\mu, \mathbb{P}) = \mathbb{E}^{\mathbb{P}} \left(\int_0^T \int_U f(t, X_t, \mu_t, u)\, Q_t(du)dt \right).$$

For each fixed n and μ, denote by $\mathcal{R}^{[n]*}(\mu)$ the set of all the optimal control rules. We can still check that

$$\inf_{\text{relaxed control } r^n} J^{[n]}(\mu, r^n) = \inf_{\mathbb{P} \in \mathcal{R}^{[n]}(\mu)} J^{[n]}(\mu, \mathbb{P}),$$

which implies we can still restrict ourselves to control rules in analyzing $\mathbf{MFG}^{[n]}$.

The proof of the following theorem is very similar to that of Theorem 3.2.11 and is hence omitted.

Theorem 3.3.3. *Suppose $\mathcal{A}_1$-$\mathcal{A}_6$ hold. For each n, there exists a relaxed solution $\mathbb{P}^{[n]}$ to $\boldsymbol{MFG}^{[n]}$.*

By Proposition 3.2.1, the sequence $\{\mathbb{P}^{[n]}\}_{n \geq 1}$ is relatively compact. Denote its limit (up to a subsequence) by $\mathbb{P}^*$ and set $\mu^* = \mathbb{P}^* \circ X^{-1}$. Then, μ^* is the limit of $\mu^{[n]} := \mathbb{P}^{[n]} \circ X^{-1}$. The following lemma shows that $\mathbb{P}^*$ is admissible.

Lemma 3.3.4. *Suppose $\mathcal{A}_1$-$\mathcal{A}_2$, $\mathcal{A}_4$-$\mathcal{A}_6$ hold. Then $\mathbb{P}^* \in \mathcal{R}(\mu^*)$.*

Proof. By Proposition 3.2.5 there exists, for each n, a $\{\mathcal{F}_t, 0 \leq t \leq T + \epsilon\}$ adapted continuous process Y^n, such that

$$\mathbb{P}^{[n]} \left(X_t = Y_t^n + \int_0^t c(s)\, dZ_s^{[n]},\ t \in [0, T+\epsilon] \right) = 1.$$

Arguing as in the proof of Proposition 3.2.6, there exists a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \mathbb{Q})$ supporting random varibales $(\widetilde{X}^n, \widetilde{Y}^n, \widetilde{Q}^n, \widetilde{Z}^n)$ and $(\widetilde{X}, \widetilde{Y}, \widetilde{Q}, \widetilde{Z})$ such that

$$(\widetilde{X}^n, \widetilde{Y}^n, \widetilde{Q}^n, \widetilde{Z}^n) \to (\widetilde{X}, \widetilde{Y}, \widetilde{Q}, \widetilde{Z}) \ \mathbb{Q}\text{-a.s.}$$

and

$$\mathbb{P}^{[n]} \circ (X, Y^n, Q, Z)^{-1} = \mathbb{Q} \circ (\widetilde{X}^n, \widetilde{Y}^n, \widetilde{Q}^n, \widetilde{Z}^n)^{-1},$$

which implies

$$\mathbb{Q} \left(\widetilde{X}_t^n = \widetilde{Y}_t^n + \int_0^t c(s)\, d\widetilde{Z}_s^{[n],n},\ t \in [0, T+\epsilon] \right) = 1, \tag{3.43}$$

where $\widetilde{Z}_t^{[n],n} = n\int_{(t-1/n)}^t \widetilde{Z}_s^n\,ds$. For each fixed $\widetilde{\omega} \in \widetilde{\Omega}$ and for each t which is a continuous point of $\widetilde{Z}(\widetilde{\omega})$, by (A.14) in Proposition A.4.1, we have

$$\begin{aligned}\left| n\int_{t-\frac{1}{n}}^t \widetilde{Z}_s^n(\widetilde{\omega})\,ds - \widetilde{Z}_t(\widetilde{\omega})\right| &\leq n\int_{t-\frac{1}{n}}^t |\widetilde{Z}_s^n(\widetilde{\omega}) - \widetilde{Z}_s(\widetilde{\omega})|\,ds + n\int_{t-\frac{1}{n}}^t |\widetilde{Z}_s(\widetilde{\omega}) - \widetilde{Z}_t(\widetilde{\omega})|\,ds \\ &\leq \sup_{t-\frac{1}{n}\leq s\leq t} |\widetilde{Z}_s^n(\widetilde{\omega}) - \widetilde{Z}_s(\widetilde{\omega})| + \sup_{t-\frac{1}{n}\leq s\leq t} |\widetilde{Z}_s(\widetilde{\omega}) - \widetilde{Z}_t(\widetilde{\omega})| \\ &\to 0.\end{aligned}$$

Then (3.43) and right-continuity of the path yield that

$$\mathbb{Q}\left(\widetilde{X}_t = \widetilde{Y}_t + \int_0^t c(s)\,d\widetilde{Z}_s,\ t \in [0, T+\epsilon]\right) = 1. \tag{3.44}$$

The desired result can be obtained by the same proof as Proposition 3.2.6.

□

Remark 3.3.5. In the above proof, the local uniform convergence near a continuous point is necessary. As stated in Proposition A.4.1, this is a direct consequence of the convergence in the M_1 topology. Local uniform convergence cannot be guaranteed in the Meyer-Zheng topology. For Meyer-Zheng topology, we only know that convergence is equivalent to convergence in Lebesgue measure but we do not have uniform convergence in general.

We are now ready to state and prove the main result of this section.

Theorem 3.3.6. *Suppose $\mathcal{A}_1$-$\mathcal{A}_6$ hold. Then $\mathbb{P}^*$ is a relaxed solution to the MFG* (3.39).

Proof. For each $\mathbb{P} \in \mathcal{R}(\mu^*)$ such that $J(\mu^*, \mathbb{P}) < \infty$, on an extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \{\widetilde{\mathcal{F}}_t, t \in \mathbb{R}\}, \widetilde{\mathbb{P}})$ we have,

$$d\widetilde{X}_t = \int_U b(t, \widetilde{X}_t, \mu_t^*, u)\,\widetilde{Q}_t(du)dt + \int_U \sigma(t, \widetilde{X}_t, \mu_t^*, u)\,\widetilde{M}(du, dt) + c(t)\,d\widetilde{Z}_t,$$

and $\mathbb{P} = \widetilde{\mathbb{P}} \circ (\widetilde{X}, \widetilde{Q}, \widetilde{Z})^{-1}$. Let $\widetilde{Z}_t^{[n]} = n\int_{t-1/n}^t \widetilde{Z}_s\,ds$. By the Lipschitz continuity of the coefficient b and σ, there exists a unique strong solution X^n to the following SDE on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \{\widetilde{\mathcal{F}}_t, t \in \mathbb{R}\}, \widetilde{\mathbb{P}})$:

$$dX_t^n = \int_U b(t, X_t^n, \mu_t^{[n]}, u)\,\widetilde{Q}_t(du)dt + \int_U \sigma(t, X_t^n, \mu_t^{[n]}, u)\,\widetilde{M}(du, dt) + c(t)\,d\widetilde{Z}_t^{[n]}.$$

For each n, set $\mathbb{P}^n = \widetilde{\mathbb{P}} \circ (X^n, \widetilde{Q}, \widetilde{Z})^{-1}$. It is easy to check that $\mathbb{P}^n \in \mathcal{R}^{[n]}(\mu^{[n]})$.

Standard estimates yield,

$$
\begin{aligned}
&\mathbb{E}^{\widetilde{\mathbb{P}}} \int_0^T |X_t^n - \widetilde{X}_t|^2\, dt \\
\le\, & C\mathbb{E}^{\widetilde{\mathbb{P}}} \int_0^T \left| \int_0^t c(s)\, d\widetilde{Z}_s^{[n]} - \int_0^t c(s)\, d\widetilde{Z}_s \right|^2 dt \\
&+\ C\mathbb{E}^{\widetilde{\mathbb{P}}} \int_0^T \Big(1 + L(W_p(\mu_t^{[n]}, \delta_0), W_p(\mu_t^*, \delta_0))\Big)^2 \mathcal{W}_p(\mu_t^{[n]}, \mu_t^*)^2\, dt.
\end{aligned}
\tag{3.45}
$$

$\widetilde{Z}^{[n]} \to \widetilde{Z}$ in M_1 a.s. implies

$$
\mathbb{E}^{\widetilde{\mathbb{P}}} \int_0^T \left| \int_0^t c(s)\, d\widetilde{Z}_s^{[n]} - \int_0^t c(s)\, d\widetilde{Z}_s \right|^2 dt \to 0.
$$

By the same arguments leading to (3.13) in the proof of Lemma 3.2.3,

$$
\mathbb{E}^{\widetilde{\mathbb{P}}} \int_0^T \Big(1 + L(W_p(\mu_t^{[n]}, \delta_0), W_p(\mu_t^*, \delta_0))\Big)^2 \mathcal{W}_p(\mu_t^{[n]}, \mu_t^*)^2\, dt \to 0.
$$

This yields,

$$
\lim_{n\to\infty} \mathbb{E}^{\widetilde{\mathbb{P}}} \int_0^T |X_t^n - \widetilde{X}_t|^2\, dt = 0. \tag{3.46}
$$

Hence, up to a subsequence, dominated convergence implies

$$
\begin{aligned}
\lim_{n\to\infty} J^{[n]}(\mu^{[n]}, \mathbb{P}^n) =\ & \lim_{n\to\infty} \mathbb{E}^{\widetilde{\mathbb{P}}} \left[\int_0^T \int_U f(t, X_t^n, \mu_t^{[n]}, u)\, \widetilde{Q}_t(du) dt \right] \\
=& \mathbb{E}^{\widetilde{\mathbb{P}}} \left[\int_0^T \int_U f(t, X_t, \mu_t^*, u)\, \widetilde{Q}_t(du) dt \right] \\
=& J(\mu^*, \mathbb{P}).
\end{aligned}
$$

Moreover, by Lemma 3.2.3,

$$
\lim_{n\to\infty} J^{[n]}(\mu^{[n]}, \mathbb{P}^{[n]}) = J(\mu^*, \mathbb{P}^*).
$$

Altogether, this yields,

$$
J(\mu^*, \mathbb{P}) = \lim_{n\to\infty} J^{[n]}(\mu^{[n]}, \mathbb{P}^n) \ge \lim_{n\to\infty} J^{[n]}(\mu^{[n]}, \mathbb{P}^{[n]}) = J(\mu^*, \mathbb{P}^*).
$$

□

3.3.2. Approximating a given solutions to MFGs with singular controls

In this subsection, we show how to approximate a *given* solution to an MFG with singular controls of the form (3.39) introduced in the previous subsection by a sequence of admissible control rules of MFGs with only regular controls.

Let $\mathbb{P}^*$ be any solution to the MFG (3.39). Since $(\Omega, \{\mathcal{F}_t, t \in \mathbb{R}\}, \mathbb{P}^*, X, Q, Z)$ satisfies the associated martingale problem, there exists a tuple $(\widehat{X}, \widehat{Q}, \widehat{Z}, M)$ defined on some extension $(\widehat{\Omega}, \{\widehat{\mathcal{F}}_t, t \in \mathbb{R}\}, \mathbb{Q})$ of the canonical path space, such that

$$\mathbb{P}^* \circ (X, Q, Z)^{-1} = \mathbb{Q} \circ (\widehat{X}, \widehat{Q}, \widehat{Z})^{-1}$$

and

$$\begin{aligned} \mathbb{Q}\Bigg(\widehat{X}_\cdot = &\int_0^\cdot \int_U b(s, \widehat{X}_s, \mu_s^*, u)\, \widehat{Q}_s(du)ds \\ &+ \int_0^\cdot \int_U \sigma(s, \widehat{X}, \mu_s^*, u)\, M(du, ds) + \int_0^\cdot c(s)\, d\widehat{Z}_s\Bigg) = 1. \end{aligned} \tag{3.47}$$

Let $X^{[n]}$ be the unique strong solution of the SDE

$$dX_t^{[n]} = \int_U b(t, X_t^{[n]}, \mu_t^{[n]}, u)\, \widehat{Q}_t(du)dt + \int_U \sigma(t, X_t^{[n]}, \mu_t^{[n]}, u)\, M(du, dt) + c(t)\, d\widehat{Z}_t^{[n]}, \tag{3.48}$$

where $\widehat{Z}^{[n]}$ is defined by (3.40) and $\mu^{[n]}$ is any sequence satisfying $\mu^{[n]} \to \mu^*$ in $\mathcal{W}_{p,(\widetilde{\mathcal{D}}(\mathbb{R}), d_{M_1})}$. One checks immediately that

$$\mathbb{P}^{[n]} := \mathbb{Q} \circ (X^{[n]}, \widehat{Q}, \widehat{Z})^{-1} \in \mathcal{R}^{[n]}(\mu^{[n]}).$$

Our goal is to show that the sequence $\{\mathbb{P}^{[n]}\}_{n \geq 1}$ converges to $\mathbb{P}^*$ in $\mathcal{W}_p$ along some subsequence, which relies on the following lemma. Its proof uses the notion of a parameter representation of the thin graph of a function $x \in \mathcal{D}(0, T)$ introduced in Appendix A.4.

Proposition 3.3.7. *On some probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{P})$, let X^n and X be the unique strong solution to SDE,*

$$dX_t^n = \int_U b(t, X_t^n, \mu_t^n, u)\, Q_t(du)dt + \int_U \sigma(t, X_t^n, \mu_t^n, u)\, M(du, dt) + dZ_t^n, \ \ t \in [0, \widetilde{T}] \tag{3.49}$$

respectively,

$$dX_t = \int_U b(t, X_t, \mu_t, u)\, Q_t(du)dt + \int_U \sigma(t, X_t, \mu_t, u)\, M(du, dt) + dZ_t, \ \ t \in [0, \widetilde{T}] \tag{3.50}$$

where $\widetilde{T}$ is a fixed positive constant, b and σ satisfy $\mathcal{A}_1$ and $\mathcal{A}_5$. If $Z^n \to Z$ in $(\mathcal{A}^m(0, \widetilde{T}), d_{M_1})$ a.s. and $\mu^n \to \mu$ in $\mathcal{W}_{p,(\mathcal{D}(0,\widetilde{T}), d_{M_1})}$, then

$$\lim_{n \to \infty} E^{\mathbb{P}} d_{M_1}(X^n, X) = 0.$$

Proof. By the a.s. convergence of Z^n to Z in M_1, there exists $\underline{\Omega} \subseteq \Omega$ with full measure such that $d_{M_1}(Z^n(\omega), Z(\omega)) \to 0$ for each $\omega \in \underline{\Omega}$. Furthermore, by Proposition

A.4.1(2), for each $\omega \in \underline{\Omega}$, there exist parameter representations $(u(\omega), r(\omega)) \in \Pi_{Z(\omega)}$ and $(u_n(\omega), r_n(\omega)) \in \Pi_{Z^n(\omega)}$ of $Z(\omega)$ and $Z^n(\omega)$ $(n \in \mathbb{N})$, respectively, such that

$$\|u_n(\omega) - u(\omega)\| \to 0 \text{ and } \|r_n(\omega) - r(\omega)\| \to 0. \tag{3.51}$$

Parameter representations with the desired convergence properties are constructed in, e.g., [PW10, Section 4]; see also [PW10, Theorem 1.2]. A careful inspection of [PW10, Section 4] shows that the constructions of $(u(\omega), r(\omega))$ and $(u_n(\omega), r_n(\omega))$ only use measurable operations. As a result the mappings $(u(\cdot), r(\cdot))$ and $(u_n(\cdot), r_n(\cdot))$ are measurable.

We now construct parameter representations $(u_{X^n}(\omega), r_{X^n}(\omega))$ and $(u_X(\omega), r_X(\omega))$ of $X^n(\omega)$ and $X(\omega)$, respectively. Since $X(\omega)$ (resp. $X^n(\omega)$) jumps at the same time as $Z(\omega)$ (resp. $Z^n(\omega)$), we can choose

$$r_X(\omega) = r(\omega), \quad r_{X^n}(\omega) = r_n(\omega).$$

In the following, we will drop the dependence on $\omega \in \underline{\Omega}$, if there is no confusion. By [PW10, equation (3.1)], parameter representations of X^n and X in terms of the parameter representations of Z^n and Z are given by, respectively,

$$\begin{aligned}
&u_{X^n}(t) \\
&= \int_0^{r_n(t)} \int_U b(s, X_s^n, \mu_s^n, u)\, Q_s(du) ds + \int_0^{r_n(t)} \int_U \sigma(s, X_s^n, \mu_s^n, u)\, M(du, ds) + u_n(t),
\end{aligned}$$

and

$$u_X(t) = \int_0^{r(t)} \int_U b(s, X_s, \mu_s, u)\, Q_s(du) ds + \int_0^{r(t)} \int_U \sigma(s, X_s, \mu_s, u)\, M(du, ds) + u(t).$$

Hence, by the Lipschitz property of b and σ and BDG's inequality, we get,

$$\begin{aligned}
&\mathbb{E} \sup_{0 \le t \le \widetilde{T}} |u_{X^n}(t) - u_X(t)| \le C\mathbb{E} \left(\int_0^{\widetilde{T}} |X^n(s) - X(s)|^2 \, ds \right)^{\frac{1}{2}} \\
&+ C \left(\int_0^{\widetilde{T}} (1 + L(\mathcal{W}_p(\mu_s^n, \delta_0), \mathcal{W}_p(\mu_s, \delta_0)))^2 \, \mathcal{W}_p^2(\mu_s^n, \mu_s) \, ds \right)^{\frac{1}{2}} \\
&+ \mathbb{E} \sup_{0 \le t \le \widetilde{T}} \left| \int_0^{r_n(t)} \int_U \sigma(s, X_s, \mu_s, u)\, M(du, ds) - \int_0^{r(t)} \int_U \sigma(s, X_s, \mu_s, u)\, M(du, ds) \right| \\
&+ C\mathbb{E} \sup_{0 \le t \le \widetilde{T}} |r_n(t) - r(t)| + \mathbb{E} \sup_{0 \le t \le \widetilde{T}} |u_n(t) - u(t)| .
\end{aligned} \tag{3.52}$$

The same argument as in the proof of Theorem 3.3.6 yields that the first two terms on the right hand side of (3.52) converge to 0 while the last three terms converge to 0 due to (3.51). Thus,

$$\lim_{n \to \infty} \mathbb{E} \sup_{0 \le t \le \widetilde{T}} |u_{X^n}(t) - u_X(t)| = 0.$$

$\square$

Corollary 3.3.8. *Under the assumptions of Proposition 3.3.7, along a subsequence $\mathbb{P}^{[n]} \to \mathbb{P}^*$ in $\mathcal{W}_p$.*

Proof. For each $\widetilde{\epsilon} > 0$, we extend the equations (3.47) and (3.48) by

$$\widehat{X}_s = \int_{-\widetilde{\epsilon}}^{s} \int_U \widetilde{b}(t, \widehat{X}_t, \mu_t^*, u)\, \widehat{Q}_t(du)dt + \int_{-\widetilde{\epsilon}}^{s} \int_U \widetilde{\sigma}(t, \widehat{X}_t, \mu_t^*, u)\, M(du, dt) + \int_{-\widetilde{\epsilon}}^{s} \widetilde{c}(t)\, d\widehat{Z}_t,$$

respectively,

$$\begin{aligned} X_s^{[n]} = &\int_{-\widetilde{\epsilon}}^{s} \int_U \widetilde{b}(t, X_t^{[n]}, \mu_t^{[n]}, u)\, \widehat{Q}_t(du)dt \\ &+ \int_{-\widetilde{\epsilon}}^{s} \int_U \widetilde{\sigma}(t, X_t^{[n]}, \mu_t^{[n]}, u)\, M(du, dt) + \int_{-\widetilde{\epsilon}}^{s} \widetilde{c}(t)\, d\widehat{Z}_t^{[n]}, \end{aligned}$$

where

$$\begin{aligned} &\widetilde{b}(s, \cdot) = b(s, \cdot),\ \widetilde{\sigma}(s, \cdot) = \sigma(s, \cdot),\ \widetilde{c}(s) = c(s) \text{ when } s \geq 0; \\ &\widetilde{b}(s, \cdot) = 0,\ \widetilde{\sigma}(s, \cdot) = 0,\ \widetilde{c}(s) = c(0) \text{ when } s < 0. \end{aligned}$$

Moreover, we have that

$$\int_{-\widetilde{\epsilon}}^{\cdot} \widetilde{c}(t)\, d\widehat{Z}_t^{[n]} = \int_{-\widetilde{\epsilon}}^{\cdot} \widetilde{c}^{+}(t)\, d\widehat{Z}_t^{[n]} - \int_{-\widetilde{\epsilon}}^{\cdot} \widetilde{c}^{-}(t)\, d\widehat{Z}_t^{[n]},$$

where a.s. in $(\mathcal{A}^m(-\widetilde{\epsilon}, T + \epsilon), d_{M_1})$,

$$\int_{-\widetilde{\epsilon}}^{\cdot} \widetilde{c}^{+}(t)\, d\widehat{Z}_t^{[n]} \to \int_{-\widetilde{\epsilon}}^{\cdot} \widetilde{c}^{+}(t)\, d\widehat{Z}_t \quad \text{and} \quad \int_{-\widetilde{\epsilon}}^{\cdot} \widetilde{c}^{-}(t)\, d\widehat{Z}_t^{[n]} \to \int_{-\widetilde{\epsilon}}^{\cdot} \widetilde{c}^{-}(t)\, d\widehat{Z}_t.$$

Since $\int_{-\widetilde{\epsilon}}^{\cdot} \widetilde{c}^{+}(t)\, d\widehat{Z}_t$ and $\int_{-\widetilde{\epsilon}}^{\cdot} \widetilde{c}^{-}(t)\, d\widehat{Z}_t$ never jump at the same time, Proposition A.4.8 implies that

$$\int_{-\widetilde{\epsilon}}^{\cdot} \widetilde{c}(t)\, d\widehat{Z}_t^{[n]} \to \int_{-\widetilde{\epsilon}}^{\cdot} \widetilde{c}(t)\, d\widehat{Z}_t$$

a.s. in $(\mathcal{A}^m(-\widetilde{\epsilon}, T + \epsilon), d_{M_1})$. Hence, by Proposition 3.3.7,

$$\mathbb{E}^{\mathbb{Q}} d_{M_1}(X^{[n]}, \widehat{X}) \to 0.$$

Hence, up to a subsequence,

$$d_{M_1}(X^{[n]}, \widehat{X}) \to 0 \text{ in } \mathcal{D}(-\widetilde{\epsilon}, T + \epsilon); \quad \mathbb{Q}\text{-a.s.},$$

which implies the same convergence holds in $\widetilde{\mathcal{D}}_{0,T+\epsilon}(\mathbb{R})$. For any nonnegative continuous function ϕ satisfying

$$\phi(x, q, z) \leq C(1 + d_{M_1}(x, 0)^p + \mathcal{W}_p^p(q/T, \delta_0) + d_{M_1}(z, 0)^p),$$

the uniform integrability of $d_{M_1}(X^{[n]}, 0)^p$, $\mathcal{W}_p^p(\widehat{Q}/T, \delta_0)$ and $d_{M_1}(\widehat{Z}, 0)^p$ yields

$$\mathbb{E}^{\mathbb{Q}} \phi(X^{[n]}, \widehat{Q}, \widehat{Z}) \to \mathbb{E}^{\mathbb{Q}} \phi(\widehat{X}, \widehat{Q}, \widehat{Z}).$$

This implies $\mathbb{Q} \circ (X^{[n]}, \widehat{Q}, \widehat{Z})^{-1} \to \mathbb{Q} \circ (\widehat{X}, \widehat{Q}, \widehat{Z})^{-1}$ in $\mathcal{W}_{p,\Omega}$ by [Vil09, Definition 6.8], that is, $\mathbb{P}^{[n]} \to \mathbb{P}^*$ in $\mathcal{W}_{p,\Omega}$. $\square$

4. PART II-2: A Mean Field Game of Optimal Portfolio Liquidation

Let $(\Omega, \mathcal{G}, \{\mathcal{G}_t, t \geq 0\}, \mathbb{P})$ be a probability space that carries independent standard Brownian motions $W^0, W^1, ..., W^N$. We consider a game of optimal portfolio liquidation with asymmetric information between a large number N of players. Following [CL17] we assume that the transaction price for each player $i = 1, ..., N$ is

$$S_t^i = \sigma W_t^0 - \int_0^t \frac{\kappa_s^i}{N} \sum_{j=1}^N \xi_s^j \, ds - \eta_t^i \xi_t^i$$

where W^0 is a standard Brownian motion. In particular, the permanent price impact depends on the players' average trading rate. The optimization problem of player $i = 1, ..., N$ is thus to minimize the cost functional

$$J^i(\xi) = \mathbb{E}\left[\int_0^T \left(\frac{\kappa_t^i}{N} \sum_{j=1}^N \xi_t^j X_t^i + \eta_t^i (\xi_t^i)^2 + \lambda_t^i (X_t^i)^2\right) dt\right] \tag{4.1}$$

subject to the state dynamics

$$dX_t^i = -\xi_t^i \, dt, \;\; X_0^i = x^i \text{ and } X_T^i = 0. \tag{4.2}$$

Here, $\xi = (\xi^1, \cdots, \xi^N)$ is the vector of strategies of each player, and κ^i, η^i and λ^i are progressively measurable with respect to the σ-field

$$\mathbb{F}^i := (\mathcal{F}_t^i, 0 \leq t \leq T), \quad \text{with} \quad \mathcal{F}_t^i := \sigma(W_s^0, W_s^i, 0 \leq s \leq t).$$

We prove the existence of approximate Nash equilibria for large populations by an MFG approach. Hence, the MFG associated with the N player game (4.1) and (4.2) is given by:

$$\begin{cases} \text{1. fix a } \mathbb{F}^0 \text{ progressively measurable process } \mu \text{ (in some suitable space);} \\ \text{2. solve the corresponding parameterized constrained optimization problem :} \\ \inf_\xi \mathbb{E}\left[\int_0^T \left(\kappa_s \mu_s X_s + \eta_s \xi_s^2 + \lambda_s X_s^2\right) ds\right] \\ \;\text{s.t. } dX_t = -\xi_t \, dt, \;\; X_0 = x \text{ and } X_T = 0; \\ \text{3. search for the fixed point } \mu_t = \mathbb{E}[\xi_t^* | \mathcal{F}_t^0], \text{ for } a.e. \; t \in [0, T], \\ \;\text{where } \xi^* \text{ is the optimal strategy from 2.} \end{cases} \tag{4.3}$$

Here, $\mathbb{F}^0 := (\mathcal{F}_t^0, 0 \leq t \leq T)$ with $\mathcal{F}_t^0 = \sigma(W_s^0, 0 \leq s \leq t)$ and κ, η and λ are $\mathbb{F} := (\mathcal{F}_t, 0 \leq t \leq T)$ progressively measurable with $\mathcal{F}_t := \sigma(W_s^0, W_s, 0 \leq s \leq t)$, where W^0 and W are independent Brownian motions of 1 and $m-1$ dimension, respectively, defined on some filtered probability space $(\Omega, \mathcal{G}, \mathbb{P})$.

The remainder of this chapter is organized as follows. In Section 4.1 we state and prove our existence and uniqueness of solutions result for the MFG (4.3). In a first step we prove that the adjoint equation associated with the MFG (4.3) has a unique solution. Then, we verify that the adjoint equation does indeed yield the optimal solution. Subsequently we prove that the solution to the MFG yields an ϵ-Nash equilibrium in a game with finitely many player and provide an explicit solution to a deterministic benchmark model. In Section 4.2 we prove that the MFG with singular terminal condition can be approximated by MFGs that penalize open positions at the terminal time.

Notation. Throughout, we adopt the convention that C denotes a constant which may vary from line to line. Moreover, for a filtration $\mathbb{G}$, $\mathrm{Prog}(\mathbb{G})$ denotes the sigma-field of progressive subsets of $[0,T]\times\Omega$ and we consider the set of progressively measurable processes w.r.t. $\mathbb{G}$:

$$\mathcal{P}_{\mathbb{G}}([0,T]\times\Omega;\mathbb{I}) = \{u : [0,T]\times\Omega \to \mathbb{I} \mid u \text{ is } \mathrm{Prog}(\mathbb{G}) - \text{measurable}\}.$$

We define the following subspaces of $\mathcal{P}_{\mathbb{G}}([0,T]\times\Omega;\mathbb{I})$:

$$\begin{aligned}
L^{\infty}_{\mathbb{G}}([0,T]\times\Omega;\mathbb{I}) &= \left\{u \in \mathcal{P}_{\mathbb{G}}([0,T]\times\Omega;\mathbb{I});\ \operatorname*{ess\,sup}_{t,\omega} |u(t,\omega)| < \infty\right\};\\
L^{p}_{\mathbb{G}}([0,T]\times\Omega;\mathbb{I}) &= \left\{u \in \mathcal{P}_{\mathbb{G}}([0,T]\times\Omega;\mathbb{I});\ \mathbb{E}\left(\int_0^T |u(t,\omega)|^2 dt\right)^{p/2} < \infty\right\}.
\end{aligned}$$

4.1. Probabilistic approach to MFGs with state constraint

In this section, we state and prove an existence and uniqueness of solutions result for the MFG (4.3). A control ξ is admissible in that game if $\xi \in \mathcal{A}_{\mathbb{F}}(t,x)$ with

$$\mathcal{A}_{\mathbb{F}}(t,x) = \left\{\xi \in L^2_{\mathbb{F}}([t,T]\times\Omega),\ \int_t^T \xi_s\, ds = x\right\}.$$

Thus, it is reasonable to fix $\mu \in L^2_{\mathbb{F}^0}([0,T]\times\Omega;\mathbb{R})$. We denote the value function of the resulting optimization problem as

$$V(t,x;\mu) := \inf_{\xi\in\mathcal{A}_{\mathbb{F}}(t,x)} \mathbb{E}\left[\int_t^T \left(\kappa_s X_s \mu_s + \eta_s \xi_s^2 + \lambda_s X_s^2\right) ds \middle| \mathcal{F}_t\right].$$

Denote by Y the adjoint process to X. The corresponding Hamiltonian to the optimization problem is

$$H(t,\xi,X,Y;\mu) = -\xi Y + \kappa_t \mu X + \eta_t \xi^2 + \lambda_t X^2.$$

By the stochastic Pontryagin maximum principle, the optimization problem reduces to the following FBSDE:

$$\begin{cases} dX_t = & -\,\xi_t\,dt, \\ -dY_t = & (\kappa_t\mu_t + 2\lambda_t X_t)\;dt - Z_t\,d\widetilde{W}_t, \\ X_0 = & x \\ X_T = & 0, \end{cases} \tag{4.4}$$

where $\widetilde{W} = (W^0, W)$ is a m-dimensional Brownian motion. The liquidation constraint $X_T = 0$ results in the singularity of the value function at liquidation time; see [GHS17]. As a result, the terminal condition for Y cannot be determined a priori. It is implicitly encoded in the FBSDE (4.4).

A standard approach yields the candidate optimal control

$$\xi_t^* = \frac{Y_t}{2\eta_t}. \tag{4.5}$$

Thus, the probabilistic method to MFGs reduces the analysis of the MFG to the analysis of the following conditional mean-field type FBSDE

$$\begin{cases} dX_t = & -\,\dfrac{Y_t}{2\eta_t}\,dt, \\ -dY_t = & \left(\kappa_t\mathbb{E}\left[\dfrac{Y_t}{2\eta_t}\,\middle|\,\mathcal{F}_t^0\right] + 2\lambda_t X_t\right)\,dt - Z_t\,d\widetilde{W}_t, \\ X_0 = & x \\ X_T = & 0. \end{cases} \tag{4.6}$$

To construct a solution to the problem (4.6), we define the following weighted spaces.

Definition 4.1.1. For $\gamma \in \mathbb{R}$, the space

$$\mathcal{H}_\gamma := \{Y \in \mathcal{P}_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R}\cup\{\infty\}) :\; (T-.)^{-\gamma}Y_. \in L^\infty_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R}\cup\{\infty\})\}$$

is endowed with the norm $\|\cdot\|_{\mathcal{H}_\gamma}$

$$\|Y\|_{\mathcal{H}_\gamma} := \|Y\|_\gamma := \operatorname*{ess\,sup}_{(\omega,t)\in\Omega\times[0,T]} \{(T-t)^{-\gamma}|Y_t|\}.$$

We make the following assumption on the cost coefficients.

Assumption 4.1.2. The processes κ, λ, η and $1/\eta$ belong to $L^\infty_{\mathbb{F}}([0,T]\times\Omega;[0,\infty))$.

We denote by $\|\lambda\|$, $\|\kappa\|$, $\|\eta\|$ the bounds of the respective cost coefficients and by $\eta_\star$ the lower bound of η. The quantity,

$$\alpha = \eta_\star/\|\eta\| \in (0,1]$$

will be important for our subsequent analysis. The following is our first major result. The proof is given in the next subsection.

Theorem 4.1.3. *There exists a unique solution to the FBSDE* (4.6)*. Moreover, the MFG* (4.3) *admits a unique equilibrium* μ^**; it is given by* $\mu_t^* = \mathbb{E}\left[\xi_t^* \middle| \mathcal{F}_t^0\right]$*, a.s. a.e., where* ξ^* *is the optimal trading rate.*

4.1.1. Sovability and verification

Decoupling (4.6) by $Y = AX + B$, we obtain the following system of Riccati type equations:

$$\begin{cases} -dA_t = \left(2\lambda_t - \dfrac{A_t^2}{2\eta_t}\right) dt - Z_t^A \, d\widetilde{W}_t, \\ -dB_t = \left(\kappa_t \mathbb{E}\left[\dfrac{1}{2\eta_t}\left(A_t X_t + B_t\right) \middle| \mathcal{F}_t^0\right] - \dfrac{A_t B_t}{2\eta_t}\right) dt - Z_t^B \, d\widetilde{W}_t, \\ A_T = \infty \\ B_T = 0. \end{cases} \tag{4.7}$$

The solvability of the first Riccati equation is due to Lemma A.6.1. Moreover, A belongs to $\mathcal{H}_{-1}$. Hence the system to solve becomes:

$$\begin{cases} dX_t = -\dfrac{1}{2\eta_t}(A_t X_t + B_t)\, dt, \\ -dB_t = \left(\kappa_t \mathbb{E}\left[\dfrac{1}{2\eta_t}\left(A_t X_t + B_t\right) \middle| \mathcal{F}_t^0\right] - \dfrac{A_t B_t}{2\eta_t}\right) dt - Z_t^B \, d\widetilde{W}_t, \\ X_0 = x \\ B_T = 0. \end{cases} \tag{4.8}$$

We apply a fixed point argument to prove the existence and uniqueness. First, we prove that the process Z^B is BMO, in the sense that the martingale $M_t = \int_0^t Z_s^B d\widetilde{W}_s$ is a BMO-martingale.[1] We denote by $\|Z^B\|_{BMO}$ the BMO-norm of the related martingale M.

Lemma 4.1.4. *Assume that there exists a solution* (X, B, Z) *to* (4.8) *such that* $(X, B) \in \mathcal{H}_\alpha \times \mathcal{H}_\alpha$*. Then,* Z^B *is BMO and there exists a constant* C *such that*

$$\|Z^B\|_{BMO} \leq C(\|B\|_\alpha + \|X\|_\alpha).$$

In particular Z^B *belongs to* $L^p_{\mathbb{F}}([0, T] \times \Omega)$ *for any* $p \geq 1$*.*

Proof. By $(X, B) \in \mathcal{H}_\alpha \times \mathcal{H}_\alpha$ and $A \in \mathcal{H}_{-1}$, we get

$$\begin{aligned} &\left|(2\eta_s)^{-1} A_s B_s - \kappa_s \mathbb{E}[(2\eta_s)^{-1} A_s X_s + (2\eta_s)^{-1} B_s \middle| \mathcal{F}_s^0]\right| \\ \leq \; &\left[\frac{1}{2\eta_\star}\|B\|_\alpha + \frac{\|\kappa\|}{2\eta_\star}\left(\|X\|_\alpha + T\|B\|_\alpha\right)\right] (T - s)^{\alpha - 1}. \end{aligned}$$

[1]For the definition of BMO-martingale and their properties we refer to the book [Kaz94].

Now, since

$$
\begin{aligned}
&\int_t^T Z_s^B d\widetilde{W}_s \\
&= B_t + \int_t^T \{(2\eta_s)^{-1}A_sB_s - \kappa_s\mathbb{E}[(2\eta_s)^{-1}A_sX_s + (2\eta_s)^{-1}B_s \big| \mathcal{F}_s^0]\}\, ds
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\left|\int_t^T Z_s^B d\widetilde{W}_s\right| \\
&\leq |B_t| + C\left(\|B\|_\alpha + \|X\|_\alpha\right)\int_t^T (T-s)^{\alpha-1} ds \\
&\leq C(T-t)^\alpha\left(\|B\|_\alpha + \|X\|_\alpha\right).
\end{aligned}
$$

Hence, the martingale $M_t = \int_0^t Z_s^B d\widetilde{W}_s$ is a BMO-martingale. □

Next, we prove an existence of solutions result for the FBSDE (4.8).

Theorem 4.1.5. *There exists $T_1 > 0$ such that when $T \leq T_1$, the FBSDE* (4.8) *admits a unique solution $(X, B, Z^B) \in \mathcal{H}_\alpha \times \mathcal{H}_\alpha \times L^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R}^m)$ to the system* (4.8)*, and the martingale $\int_0^\cdot Z_s^B\, d\widetilde{W}_s$ is a BMO-martingale.*

Proof. For any $(x, b) \in \mathcal{H}_\alpha \times \mathcal{H}_\alpha$, we introduce the following FBSDE

$$
\begin{cases}
dX_t = -\dfrac{1}{2\eta_t}(A_tX_t + b_t)\, dt, \\
-dB_t = \left(\kappa_t\mathbb{E}\left[\dfrac{1}{2\eta_t}\left(A_tX_t + b_t\right)\Big| \mathcal{F}_t^0\right] - \dfrac{A_tB_t}{2\eta_t}\right) dt - Z_t^B\, d\widetilde{W}_t, \\
X_0 = x \\
B_T = 0,
\end{cases}
\tag{4.9}
$$

which implies that

$$
X_t = xe^{-\int_0^t \frac{A_r}{2\eta_r}\, dr} - \int_0^t \frac{b_s}{2\eta_s} e^{-\int_s^t \frac{A_r}{2\eta_r}\, dr}\, ds \tag{4.10}
$$

and

$$
B_t = \mathbb{E}\left[\int_t^T \kappa_s\mathbb{E}\left[\frac{A_sX_s + b_s}{2\eta_s}\bigg| \mathcal{F}_s^0\right] e^{-\int_t^s \frac{A_r}{2\eta_r}\, dr}\, ds \bigg| \mathcal{F}_t\right]. \tag{4.11}
$$

By Lemma A.6.1, (4.10) and (4.11) yield that

$$
|X_t| \leq \frac{x(T-t)^\alpha}{T^\alpha} + \frac{1}{2\underline{\eta}}\int_0^t \frac{|b_s|}{(T-s)^\alpha}\, ds(T-t)^\alpha \tag{4.12}
$$

and

$$
\begin{aligned}
|B_t| &\leq C\mathbb{E}\left[\int_t^T \left(\frac{\mathbb{E}[|X_s||\mathcal{F}_s^0]}{T-s} + \mathbb{E}[|b_s||\mathcal{F}_s^0]\right) ds \middle| \mathcal{F}_t\right] \\
&\leq C\|X\|_\alpha (T-t)^\alpha + C\|b\|_\alpha (T-t)^\alpha.
\end{aligned}
\tag{4.13}
$$

Thus, we have $(X, B) \in \mathcal{H}_\alpha \times \mathcal{H}_\alpha$. So it defines a mapping from $\mathcal{H}_\alpha \times \mathcal{H}_\alpha$ to itself:

$$
\Phi : (x, b) \to (X, B).
$$

Now it is sufficient to prove Φ is a contraction. From (4.10), we have

$$
|X_t - X'_t| \leq C(T-t)^\alpha \int_0^t \frac{|b_s - b'_s|}{(T-s)^\alpha}\, ds \tag{4.14}
$$

and

$$
\|X - X'\|_\alpha \leq CT\|b - b'\|_\alpha. \tag{4.15}
$$

From (4.11) and (4.14), we have

$$
\begin{aligned}
|B_t - B'_t| &\leq C\mathbb{E}\left[\int_t^T \left(\frac{\mathbb{E}[|X_s - X'_s||\mathcal{F}_s^0]}{T-s} + \mathbb{E}[|b_s - b'_s||\mathcal{F}_s^0]\right) ds \middle| \mathcal{F}_t\right] \\
&\leq C\mathbb{E}\left[\int_t^T (T-s)^{\alpha-1}\mathbb{E}\left[\int_0^s \frac{|b_r - b'_r|}{(T-r)^\alpha}\, dr \middle| \mathcal{F}_s^0\right] + \mathbb{E}[|b_s - b'_s||\mathcal{F}_s^0]\, ds \middle| \mathcal{F}_t\right] \\
&\leq CT(T-t)^\alpha \|b - b'\|_\alpha,
\end{aligned}
\tag{4.16}
$$

which together with (4.15) yields that

$$
\|X - X'\|_\alpha + \|B - B'\|_\alpha \leq CT(\|x - x'\|_\alpha + \|b - b'\|_\alpha).
$$

Moreover from Lemma 4.1.4, $Z^B - (Z^B)'$ is BMO and the BMO-norm is bounded by $C(\|X - X'\|_\alpha + \|B - B'\|_\alpha)$. This shows that Φ is a contraction on $\mathcal{H}_\alpha \times \mathcal{H}_\alpha \times L^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R}^m)$ if T is small. □

As a by-product of the proof of Theorem 4.1.5, we obtain the following bounds.

Lemma 4.1.6. *There exists a constant C depending on $\|A\|_{-1}$, T, α, $\|\kappa\|$ and $|x|$, such that*

$$
\|X\|_\alpha + \|B\|_\alpha + \|Z^B\|_{BMO} \leq C.
$$

From (4.5), we get candidates of optimal position and optimal trading rate

$$
\begin{aligned}
X_t^* &= xe^{-\int_0^t \frac{A_r}{2\eta_r}\, dr} - \int_0^t \frac{B_s}{2\eta_s} e^{-\int_s^t \frac{A_r}{2\eta_r}\, dr}\, ds, \\
\xi_t^* &= xe^{-\int_0^t \frac{A_r}{2\eta_r}\, dr} \frac{A_t}{2\eta_t} + \frac{B_t}{2\eta_t} - \frac{A_t}{2\eta_t}\int_0^t \frac{B_s}{2\eta_s} e^{-\int_s^t \frac{A_r}{2\eta_r}\, dr}\, ds.
\end{aligned}
$$

In order to solve our MFG it remains to establish a verification result.

Theorem 4.1.7. *The process ξ^* is an admissible optimal control. Hence $\mu^* = \mathbb{E}[\xi^*|\mathcal{F}^0]$ is the solution to the MFG. Moreover, the value function is given by*

$$V(t,x;\mu^*) = \frac{1}{2}A_t x^2 + \frac{1}{2}B_t x + \frac{1}{2}\mathbb{E}\left[\int_t^T \kappa_s X_s^* \xi_s^* \, ds \middle| \mathcal{F}_t\right] \tag{4.17}$$

Proof. The verification argument is divided into the following four steps.

Step 1. $\xi^* \in \mathcal{A}_{\mathbb{F}}(0,x)$. Indeed, for any $\varepsilon > 0$, integration by part yields that

$$\begin{aligned}
&X_{T-\varepsilon}^* Y_{T-\varepsilon} = X_t^* Y_t + \int_t^{T-\varepsilon} X_s^* \, dY_s + \int_t^{T-\varepsilon} Y_s \, dX_s^* \\
&= X_t^* Y_t - \int_t^{T-\varepsilon} X_s^*(\kappa_s \mu_s^* + 2\lambda_s X_s^*) \, ds + \int_t^{T-\varepsilon} X_s^* Z_s \, d\widetilde{W}_s - \int_t^{T-\varepsilon} 2\eta_s (\xi_s^*)^2 \, ds,
\end{aligned}$$

which implies that

$$\mathbb{E}\left[\int_t^{T-\varepsilon} (\kappa_s \mu_s^* X_s^* + 2\lambda_s (X_s^*)^2 + 2\eta_s(\xi_s^*)^2) \, ds \middle| \mathcal{F}_t\right] + \mathbb{E}\left[X_{T-\varepsilon}^* Y_{T-\varepsilon}\middle|\mathcal{F}_t\right] = X_t^* Y_t.$$

Moreover, by the ansatz for Y, it holds that

$$\mathbb{E}\left[X_{T-\varepsilon}^* Y_{T-\varepsilon}\middle|\mathcal{F}_t\right] = \mathbb{E}\left[A_{T-\varepsilon}(X_{T-\varepsilon}^*)^2\middle|\mathcal{F}_t\right] + \mathbb{E}\left[X_{T-\varepsilon}^* B_{T-\varepsilon}\middle|\mathcal{F}_t\right] \geq \mathbb{E}\left[X_{T-\varepsilon}^* B_{T-\varepsilon}\middle|\mathcal{F}_t\right].$$

Thus, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_t^{T-\epsilon} 2\eta_s(\xi_s^*)^2 \, ds\right] \\
&\leq \mathbb{E}\left[\int_t^{T-\epsilon} \kappa_s |\mu_s^* X_s^*| \, ds\right] + \mathbb{E}\left[\int_t^{T-\epsilon} 2\lambda_s |X_s^*|^2 \, ds\right] + \mathbb{E}\left[|X_{T-\epsilon}^* B_{T-\epsilon}||\mathcal{F}_t\right] + X_t^* Y_t.
\end{aligned} \tag{4.18}$$

Since $(X^*, B) \in (\mathcal{H}_\alpha)^2$, we deduce that $X^* B \in \mathcal{H}_{2\alpha}$ and that

$$|\mu_t^*| \leq \mathbb{E}\left[\frac{1}{2\eta_t}|A_t X_t^* + B_t|\middle|\mathcal{F}_t^0\right] \leq \frac{1}{2\eta_\star}(\|AX^*\|_{-1+\alpha} + T\|B\|_\alpha)(T-t)^{\alpha-1}.$$

Applying dominated convergence to $\mathbb{E}\left[X_{T-\epsilon}^* B_{T-\epsilon}|\mathcal{F}_t\right]$ and monotone convergence to the other terms in (4.18), we get

$$\mathbb{E}\left[\int_t^T 2\eta_s(\xi_s^*)^2 \, ds\right] \leq \mathbb{E}\left[\int_t^T \kappa_s |\mu_s^* X_s^*| \, ds\right] + \mathbb{E}\left[\int_t^T 2\lambda_s |X_s^*|^2 \, ds\right] + X_t^* Y_t.$$

Since $X^* \in \mathcal{H}_\alpha$ and $\mu^* \in \mathcal{H}_{\alpha-1}$, $\kappa\mu^* X^*$ is in $L^1_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R})$. Thus, we obtain

$$2\eta_\star \mathbb{E}\left[\int_0^T \xi_s^*)^2 \, ds\right] \leq \mathbb{E}\left[\int_0^T (2\eta_s(\xi_s^*)^2) \, ds\right] < +\infty.$$

By Theorem 4.1.5, $X_T^* = 0$. Admissibility of ξ^* is then proved.

Step 2. For any $\xi \in \mathcal{A}_{\mathbb{F}}(t,x)$, let X^ξ be the corresponding state process. Then it holds that:

$$\lim_{s \nearrow T} \mathbb{E}\left[X_s^\xi Y_s \middle| \mathcal{F}_t\right] = 0.$$

Indeed since $A \in \mathcal{H}_{-1}$, for any $0 \le t \le s < T$

$$\begin{aligned}
\left|\mathbb{E}\left[X_s^\xi Y_s \middle| \mathcal{F}_t\right]\right| &= \left|\mathbb{E}\left[X_s^\xi (X_s^* A_s + B_s^*) \middle| \mathcal{F}_t\right]\right| \\
&\le \frac{C}{T-s}\mathbb{E}\left[(X_s^\xi)^2 + (X_s^*)^2 \middle| \mathcal{F}_t\right] + \mathbb{E}\left[X_s^\xi B_s^* \middle| \mathcal{F}_t\right] \\
&= \frac{C}{T-s}\mathbb{E}\left[\left(\int_s^T \xi_u\, du\right)^2 + \left(\int_s^T \xi_u^*\, du\right)^2 \middle| \mathcal{F}_t\right] + \mathbb{E}\left[X_s^\xi B_s^* \middle| \mathcal{F}_t\right] \\
&\le C\mathbb{E}\left[\int_s^T \xi_u^2\, du + \int_s^T (\xi_u^*)^2\, du \middle| \mathcal{F}_t\right] + \mathbb{E}\left[X_s^\xi B_s^* \middle| \mathcal{F}_t\right] \xrightarrow{s \nearrow T} 0.
\end{aligned}$$

Step 3. Now for each $\xi \in \mathcal{A}_{\mathbb{F}}(t,x)$, let $X = X^\xi$ be the corresponding state. For each $\varepsilon > 0$ and each $t \in [0, T-\varepsilon]$, we have from the convexity of the Hamiltonian

$$\begin{aligned}
&\mathbb{E}\left[\int_t^{T-\varepsilon} \left(\kappa_s \mu_s^* X_s + \eta_s \xi_s^2 + \lambda_s X_s^2\right) ds \middle| \mathcal{F}_t\right] \\
&\quad - \mathbb{E}\left[\int_t^{T-\varepsilon} \left(\kappa_s \mu_s^* X_s^* + \eta_s (\xi_s^*)^2 + \lambda_s (X_s^*)^2\right) ds \middle| \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\int_t^{T-\varepsilon} \left(H(s,\xi_s,X_s,Y_s;\mu^*) - H(s,\xi_s^*,X_s^*,Y_s;\mu^*) + (\xi_s - \xi_s^*)Y_s\right) ds \middle| \mathcal{F}_t\right] \\
&\ge \mathbb{E}\left[\int_t^{T-\varepsilon} \left(\partial_\xi H(s,\xi_s^*,X_s^*,Y_s;\mu^*)(\xi_s - \xi_s^*)\right.\right. \\
&\quad \left.\left. + \partial_x H(s,\xi_s^*,X_s^*,Y_s;\mu^*)(X_s - X_s^*) + (\xi_s - \xi_s^*)Y_s\right) ds \middle| \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\int_t^{T-\varepsilon} \left((\kappa_s \mu_s^* + 2\lambda_s X_s^*)(X_s - X_s^*) + (\xi_s - \xi_s^*)Y_s\right) ds \middle| \mathcal{F}_t\right]
\end{aligned}$$

since $\partial_\xi H(s,\xi_s^*,X_s^*,Y_s) = 2\eta_s \xi_s^* - Y_s = 0$. Moreover, integration by part implies that for any $\varepsilon > 0$

$$\begin{aligned}
&Y_{T-\varepsilon}(X_{T-\varepsilon}^* - X_{T-\varepsilon}) \\
&= Y_t(X_t^* - X_t) + \int_t^{T-\varepsilon} (X_s^* - X_s)\, dY_s + \int_t^{T-\varepsilon} Y_s\, d(X_s^* - X_s) \\
&= -\int_t^{T-\varepsilon} (\kappa_s \mu_s^* + 2\lambda_s X_s^*)(X_s^* - X_s)\, ds + \int_t^{T-\varepsilon} Z_s (X_s^* - X_s)\, dW_s^0 \\
&\quad - \int_t^{T-\varepsilon} Y_s(\xi_s^* - \xi_s)\, ds.
\end{aligned} \tag{4.19}$$

Therefore,

$$
\begin{aligned}
&\mathbb{E}\left[\int_t^{T-\varepsilon} \left(\kappa_s\mu_s X_s + \eta_s \xi_s^2 + \lambda_s X_s^2\right)\, ds \middle| \mathcal{F}_t\right] \\
&\quad - \mathbb{E}\left[\int_t^{T-\varepsilon} \left(\kappa_s\mu_s X_s^* + \eta_s (\xi_s^*)^2 + \lambda_s (X_s^*)^2\right)\, ds \middle| \mathcal{F}_t\right] \\
&\geq \mathbb{E}\left[Y_{T-\varepsilon}(X_{T-\varepsilon}^* - X_{T-\varepsilon}) \middle| \mathcal{F}_t\right].
\end{aligned}
$$

Taking $\varepsilon \to 0$, Step 2 yields that

$$
J(t,x,\xi;\mu^*) - J(t,x,\xi^*;\mu^*) \geq 0.
$$

Hence ξ^* is an optimal control.

Step 4. In view of (4.19) and using again Step 2, we obtain that

$$
\begin{aligned}
&\mathbb{E}\left[\int_t^T (\kappa_s \xi_s^* X_s^* + \lambda_s (X_s^*)^2 + \eta_s (\xi_s^*)^2)\, ds \middle| \mathcal{F}_t\right] \\
&= \frac{1}{2} A_t (X_t^*)^2 + \frac{1}{2} B_t X_t^* + \frac{1}{2}\mathbb{E}\left[\int_t^T \kappa_s X_s^* \xi_s^*\, ds \middle| \mathcal{F}_t\right].
\end{aligned}
$$

This yields (4.17). $\square$

Remark 4.1.8. Since $(X^*, B^*) \in \mathcal{H}_\alpha \times \mathcal{H}_\alpha$ and $\xi^* \in \mathcal{A}_{\mathcal{F}}(0,x)$,

$$
\begin{aligned}
&B_t^* x + \mathbb{E}\left[\int_t^T \kappa_s \xi_s^* X_s^*\, ds \middle| \mathcal{F}_t\right] \\
&\leq \|\kappa\|(T-t)^\alpha \|X^*\|_\alpha \mathbb{E}\left[\int_0^T |\xi_s^*|\, ds \middle| \mathcal{F}_t\right] + x\|B^*\|_\alpha (T-t)^\alpha \xrightarrow{t\nearrow T} 0.
\end{aligned}
$$

As a result, we get the following terminal condition:

$$
\lim_{t\uparrow T} V(t,x;\mu) = \begin{cases} 0, & x = 0; \\ \infty, & x > 0. \end{cases}
$$

4.1.2. Approximate Nash Equilibrium

We are now going to show that solution to our MFG (4.3) yields an approximate Nash equilibrium for the N player game, when the number of players is large. In the N player game, each player $i = 1, ..., N$ chooses s strategy ξ^i to minimize the cost functional

$$
J^{N,i}(\xi^1, \cdots, \xi^N) = \mathbb{E}\left[\int_0^T \kappa_t^i \frac{1}{N}\sum_{j=1}^N \xi_t^j X_t^i + \eta_t^i (\xi_t^i)^2 + \lambda_t^i (X_t^i)^2\, dt\right],
$$

subject to the state dynamics

$$dX_t^i = -\xi_t^i \, dt, \quad X_0^i = x, \quad X_T^i = 0.$$

We assume that the coefficients κ^i, η^i and λ^i belong to $L^\infty_{\mathbb{F}^i}([0,T] \times \Omega; [0,\infty))$, where $\mathbb{F}^i := (\mathcal{F}_t^i, 0 \le t \le T)$ with $\mathcal{F}_t^i = \sigma(W_s^0, W_s^i, 0 \le s \le t)$, that $(\kappa_t^i, \eta_t^i, \lambda_t^i)$ are independent and identically distributed conditioned on $\mathcal{F}_t^0$ for any $t \in [0,T]$, and that the processes (κ^i) are uniformly bounded by $\|\kappa\|$.

Let

$$\xi^{*,i} := \frac{A^i X^{*,i} + B^{*,i}}{2\eta^i},$$

where $X^{*,i}$ and $B^{*,i}$ are solutions to the FBSDE (4.8) and A^i is the solution to the BSDE (4.7), with κ, η, λ and $\widetilde{W}$ replaced by κ^i, η^i, λ^i and (W^0, W^i), respectively. Note that $\xi^{*,1}$, $\xi^{*,2}$, $\cdots$, $\xi^{*,N}$ are independent and identically distributed, given W^0. Moreover, let

$$J^i(\xi; \mu) := \mathbb{E}\left[\int_0^T \kappa_t^i \mu_t X_t^i + \eta_t^i (\xi_t^i)^2 + \lambda_t^i (X_t^i)^2 \, dt\right].$$

The same analysis as above yields that

$$J^i(\xi; \mu^*) \ge J^i(\xi^{*,i}; \mu^*), \tag{4.20}$$

for any $\xi \in L^2_{\mathbb{F}^i}([0,T] \times \Omega; \mathbb{R})$, where

$$\mu_t^* := \mathbb{E}\left[\xi_t^{*,i} \middle| \mathcal{F}_t^0\right], \; a.e. \; t \in [0,T].$$

The following theorem shows $(\xi^{*,1}, \cdots, \xi^{*,N})$ helps construct an ϵ-Nash equilibrium for N player games when N is large.

Theorem 4.1.9. *Assume the admissible control space for player i is*

$$\begin{aligned}\mathcal{A}^i := \; & \{\xi \in \mathcal{A}_{\mathbb{F}^i}(0,x) : \\ & \mathbb{E}\left[\int_0^T |\xi_t|^2 \, dt\right] \le M \text{ such that the corresponding state } \|X\|_\alpha \le M \Big\}\end{aligned}$$

for some fixed positive constant M large enough. There exists ϵ_N with $\lim_{N\to\infty} \epsilon_N = 0$ such that for each $1 \le i \le N$,

$$J^{N,i}(\xi^{*,1}, \xi^{*,2}, \cdots, \xi^{*,N}) \le J^{N,i}(\xi^{*,-i}) + \epsilon_N,$$

where $\xi^{*,-i} = (\xi^{*,1}, \cdots, \xi^{*,i-1}, \xi^i, \xi^{*,i+1}, \cdots, \xi^{*,N})$ with $\xi^i \in \mathcal{A}^i$.

Proof. By the symmetry of the N player game, it is sufficient to show the result for $i = 1$. First, note that $\xi^{*,1} \in \mathcal{A}^1$. For each $\xi \in \mathcal{A}^1$, let X be the corresponding state process. By (4.20) we have that

$$
\begin{aligned}
& J^{N,1}(\xi, \xi^{*,2}, \cdots, \xi^{*,N}) - J^{N,1}(\xi^{*,1}, \cdots, \xi^{*,N}) \\
= & \ J^{N,1}(\xi, \xi^{*,2}, \cdots, \xi^{*,N}) - J^1(\xi; \mu^*) + J^1(\xi; \mu^*) - J^1(\xi^{*,1}; \mu^*) \\
& + J^1(\xi^{*,1}; \mu^*) - J^{N,1}(\xi^{*,1}, \cdots, \xi^{*,N}) \\
\geq & \ \mathbb{E} \int_0^T \left[\kappa_t^1 \left(\frac{1}{N} \sum_{j=2}^N \xi_t^{*,j} + \frac{1}{N} \xi_t \right) X_t + \eta_t^1 \xi_t^2 + \lambda_t^1 X_t^2 \right] dt \\
& - \mathbb{E} \left[\int_0^T \left(\kappa_t^1 \mu_t^* X_t + \eta_t^1 \xi_t^2 + \lambda_t^1 X_t^2 \right) dt \right] \\
& + \ \mathbb{E} \left[\int_0^T \left(\kappa_t^1 \mu_t^* X_t^{*,1} + \eta_t^1 (\xi_t^{*,1})^2 + \lambda_t^1 (X_t^{*,1})^2 \right) dt \right] \\
& - \ \mathbb{E} \left[\int_0^T \left(\kappa_t^1 \frac{1}{N} \sum_{j=1}^N \xi_t^{*,j} X_t^{*,1} + \eta_t^1 (\xi_t^{*,1})^2 + \lambda_t^1 (X_t^{*,1})^2 \right) dt \right] \\
:= & \ I_1 + I_2.
\end{aligned}
$$

For the first difference I_1 in the above inequality, we have

$$
\begin{aligned}
\sup_{\xi \in \mathcal{A}^1} |I_1| \leq & \ \frac{\|\kappa\|}{N} \sup_{\xi \in \mathcal{A}^1} \mathbb{E} \left[\int_0^T |X_t| |\xi_t| \, dt \right] + \|\kappa\| \mathbb{E} \left[\int_0^T |X_t| \left| \frac{1}{N} \sum_{j=2}^N \xi_t^{*,j} - \mu_t^* \right| dt \right] \\
\leq & \ \frac{M \|\kappa\| T^{\alpha + \frac{1}{2}}}{\sqrt{2\alpha} N} + \|\kappa\| M T^\alpha \mathbb{E} \left[\int_0^T \left| \frac{1}{N} \sum_{j=1}^N \xi_t^{*,j} - \mu_t^* \right| dt \right] + \frac{\|\kappa\| M^{\frac{3}{2}} T^{\alpha + \frac{1}{2}}}{N} \\
\to & \ 0.
\end{aligned}
$$

For the second difference I_2, we have

$$
\begin{aligned}
I_2 \leq & \ \|\kappa\| M T^\alpha \mathbb{E} \left[\int_0^T \left| \mu_t^* - \frac{1}{N} \sum_{j=1}^N \xi_t^{*,j} \right| dt \right] \\
\to & \ 0.
\end{aligned}
$$

If we set

$$
\begin{aligned}
\epsilon_N := & \ \frac{M \|\kappa\| T^{\alpha + \frac{1}{2}}}{\sqrt{2\alpha} N} + \|\kappa\| M T^\alpha \mathbb{E} \left[\int_0^T \left| \frac{1}{N} \sum_{j=2}^N \xi_t^{*,j} - \mu_t^* \right| dt \right] + \frac{\|\kappa\| M^{\frac{3}{2}} T^{\alpha + \frac{1}{2}}}{N} \\
& + \|\kappa\| M T^\alpha \mathbb{E} \left[\int_0^T \left| \mu_t^* - \frac{1}{N} \sum_{j=1}^N \xi_t^{*,j} \right| dt \right],
\end{aligned}
$$

then we have

$$
J^{N,1}(\xi, \xi^{*,2}, \cdots, \xi^{*,N}) - J^{N,1}(\xi^{*,1}, \cdots, \xi^{*,N}) \geq -\epsilon_N.
$$

$\square$

Remark 4.1.10. When searching for the approximate Nash equilibria, we may as well assume that the individual players have full information. That is to say, we may assume that the admissible control space for each player is

$$\mathcal{A} := \left\{ \xi \in \mathcal{A}_{\overline{\mathbb{F}}^N}(0,x) : \mathbb{E}\left[\int_0^T |\xi_t|^2\,dt\right] \leq M \text{ such that } \|X\|_\alpha \leq M \right\},$$

where $\overline{\mathbb{F}}^N = (\overline{\mathcal{F}}_t^N, 0 \leq t \leq T)$ with $\overline{\mathcal{F}}_t^N := \sigma(W_t^0, W_t^1, \cdots, W_t^N)$. By the same argument as in Section 4.1.1, we have

$$J^i(\xi;\mu^*) \geq J^i(\xi^{*,i};\mu^*),$$

for all $\xi \in \mathcal{A}_{\overline{\mathbb{F}}^N}(0,x)$. Thus, the same analysis as in Theorem 4.1.9 implies that for all $\xi \in \mathcal{A}$

$$J^{N,i}(\xi^{*,1},\xi^{*,2},\cdots,\xi^{*,N}) \leq J^{N,i}(\xi^{*,-i}) + \epsilon_N,$$

where we recall that $\xi^{*,-i} = (\xi^{*,1},\cdots,\xi^{*,i-1},\xi,\xi^{*,i+1},\cdots,\xi^{*,N})$.

4.1.3. Common Value Environment

In this section, we consider the benchmark case where all the randomness is generated by the Brownian motion W^0 that drives the benchmark price process. In particular, all players share the same information. That is, in this section we assume that the following common value environment assumption is satisfied.

Assumption 4.1.11. The processes κ, λ, η and $1/\eta$ belong to $L^\infty_{\mathbb{F}^0}([0,T]\times\Omega;[0,\infty))$.

Under the above assumption, the consistency condition reduces to

$$\mu = \xi^* \tag{4.21}$$

and the conditional mean-field FBSDE reduces to the following FBSDE

$$\begin{cases} dX_t = -\dfrac{Y_t}{2\eta_t}\,dt, \\ -dY_t = \left(\dfrac{\kappa_t Y_t}{2\eta_t} + 2\lambda_t X_t\right)dt - Z_t\,dW_t^0, \\ X_0 = x, \\ X_T = 0. \end{cases} \tag{4.22}$$

The linear ansatz $Y = AX$ yields,

$$-dA_t = \left(2\lambda_t + \frac{\kappa_t A_t}{2\eta_t} - \frac{A_t^2}{2\eta_t}\right)dt - Z_t^A dW_t^0, \quad A_T = \infty. \tag{4.23}$$

This singular terminal condition on A is necessary to satisfy the constraint $X_T = 0$. Let $\widetilde{A}_t = A_t e^{\int_0^t \frac{\kappa_s}{2\eta_s}\, ds}$. Then,

$$-d\widetilde{A}_t = \left[2\lambda_t e^{\int_0^t \frac{\kappa_s}{2\eta_s}\, ds} - \frac{\widetilde{A}_t^2}{2\eta_t e^{\int_0^t \frac{\kappa_s}{2\eta_s}\, ds}} \right] dt - \widetilde{Z}_t\, dW_t^0, \quad \widetilde{A}_T = \infty. \tag{4.24}$$

The above Riccati equation has a nonnegative solution $\widetilde{A}$, due to Lemma A.6.1. By (4.22),

$$X_t^* = x e^{-\int_0^t \frac{A_r}{2\eta_r}\, dr}.$$

Lemma 4.1.12. *Under Assumption 4.1.11, the processes A, X^*, $Y = AX^*$ and $\xi^* = \mu = Y/(2\eta)$ are all non negative and*

$$A \in \mathcal{H}_{-1}, \;\; X^* \in \mathcal{H}_\alpha, \;\; Y \in \mathcal{H}_{\alpha-1}, \;\; \xi^* \in \mathcal{H}_{\alpha-1}.$$

Proof. Due to Lemma A.6.1, the following estimate holds for any $0 \le t < T$:

$$\frac{1}{\mathbb{E}\left[\int_t^T \frac{1}{2\eta_s} e^{-\int_0^s \frac{\kappa_r}{2\eta_r}\, dr}\, ds \middle| \mathcal{F}_t^0 \right]} \le \widetilde{A}_t$$

Hence the process A_t is bounded from below by:

$$\begin{aligned} A_t \;\; &\ge \;\; \frac{e^{-\int_0^t \frac{\kappa_r}{2\eta_r}\, dr}}{\mathbb{E}\left[\int_t^T \frac{1}{2\eta_s} e^{-\int_0^s \frac{\kappa_r}{2\eta_r}\, dr}\, ds \middle| \mathcal{F}_t^0 \right]} = \frac{1}{\mathbb{E}\left[\int_t^T \frac{1}{2\eta_s} e^{-\int_t^s \frac{\kappa_r}{2\eta_r}\, dr}\, ds \middle| \mathcal{F}_t^0 \right]} \\ &\ge \;\; \frac{1}{\mathbb{E}\left[\int_t^T \frac{1}{2\eta_s}\, ds \middle| \mathcal{F}_t^0 \right]} \ge 2\eta_\star \frac{1}{(T-t)}. \end{aligned}$$

Hence,

$$e^{-\int_0^t \frac{A_r}{2\eta_r}\, dr} \le \exp\left(-2\eta_\star \int_0^t \frac{1}{2\eta_r (T-r)}\, dr \right) \le \left(\frac{T-t}{T} \right)^\alpha. \tag{4.25}$$

The conclusion on X^* can be deduced immediately. Again from Lemma A.6.1, $\widetilde{A}$ is bounded from above:

$$\widetilde{A}_t \le \frac{1}{(T-t)^2} \mathbb{E}\left[\int_t^T \left(2\eta_s e^{\int_0^s \frac{\kappa_r}{2\eta_r}\, dr} + 2(T-s)^2 \lambda_s e^{\int_0^s \frac{\kappa_r}{2\eta_r}\, dr} \right) ds \middle| \mathcal{F}_t^0 \right].$$

Thus we get an upper bound on A:

$$\begin{aligned} A_t \;\; &\le \;\; \frac{e^{-\int_0^t \frac{\kappa_r}{2\eta_r}\, dr}}{(T-t)^2} \mathbb{E}\left[\int_t^T \left(2\eta_s e^{\int_0^s \frac{\kappa_r}{2\eta_r}\, dr} + 2(T-s)^2 \lambda_s e^{\int_0^s \frac{\kappa_r}{2\eta_r}\, dr} \right) ds \middle| \mathcal{F}_t^0 \right] \\ &\le \;\; \frac{2}{(T-t)^2} \left[\|\eta\| e^{\int_0^T \frac{\kappa_r}{2\eta_r}\, dr} (T-t) + \frac{1}{3} \|\lambda\| e^{\int_0^T \frac{\kappa_r}{2\eta_r}\, dr} (T-t)^3 \right] \end{aligned}$$

$$\leq \quad \frac{2}{(T-t)} e^{\frac{\|\kappa\| T}{2\eta_\star}} \left[\|\eta\| + \frac{\|\lambda\| T^2}{3} \right].$$

Collecting all inequalities we get that $A \in \mathcal{H}_{-1}$ and

$$\begin{aligned} |\xi_t^*| = \frac{A_t |X_t^*|}{2\eta_t} &= |x| \frac{A_t e^{-\int_0^t \frac{A_r}{2\eta_r}\, dr}}{2\eta_t} \\ &\leq \frac{|x|}{\eta_\star T^\alpha} \left[\|\eta\| + \frac{\|\lambda\| T^2}{3} \right] e^{\frac{\|\kappa\| T}{2\eta_\star}} (T-t)^{\alpha - 1}. \end{aligned}$$

A similar inequality holds for Y. □

From the representation (4.22), we deduce that Y is a non negative supermartingale. In particular, the limit at the terminal time T of Y exists and is finite. Since $X^* \in \mathcal{H}_\alpha$, we deduce that $\lim_{t \nearrow T} Y_t X_t^* = 0$. Moreover the process Z belongs to $L^p_{\mathbb{F}^0}([0, T-\varepsilon] \times \Omega, [0, +\infty))$ for any $p \geq 1$ and any $\varepsilon > 0$.

The following verification theorem shows that ξ^* is optimal. The proof is similar to that of Theorem 4.1.7.

Theorem 4.1.13. $\xi^* (= \mu^*)$ *is an admissible optimal control as well as the equilibrium to MFG. Moreover the value function is given by:*

$$V(t, x; \mu^*) = \frac{1}{2} A_t x^2 + \frac{1}{2} \mathbb{E} \left[\int_t^T \kappa_s \mu_s^* X_s^* \, ds \middle| \mathcal{F}_t^0 \right]. \tag{4.26}$$

Remark 4.1.14. In the common value environment, both the optimal strategy and the optimal position are non-negative throughout the liquidation interval. We cannot prove (and do not expect) a similar result under asymmetric information.

4.1.4. An Example

In this section, we consider a deterministic benchmark example that can be solved explicitly. We assume that the following assumption holds.

Assumption 4.1.15. The processes σ, λ, κ, η are positive constants.

Under the preceding assumption, the Riccati equation (4.23) reduces to

$$-dA_t = \left(2\lambda + \frac{\kappa A_t}{2\eta} - \frac{A_t^2}{2\eta} \right) dt, \quad A_T = \infty,$$

whose explicit solution is

$$A_t = \frac{2\eta \left(\alpha_+ e^{\alpha_+ T} e^{\alpha_- t} - \alpha_- e^{\alpha_- T} e^{\alpha_+ t} \right)}{e^{\alpha_+ T} e^{\alpha_- t} - e^{\alpha_- T} e^{\alpha_+ t}},$$

where

$$\alpha_+ = \frac{\kappa + \sqrt{\kappa^2 + 16\eta\lambda}}{4\eta}, \quad \alpha_- = \frac{\kappa - \sqrt{\kappa^2 + 16\eta\lambda}}{4\eta}.$$

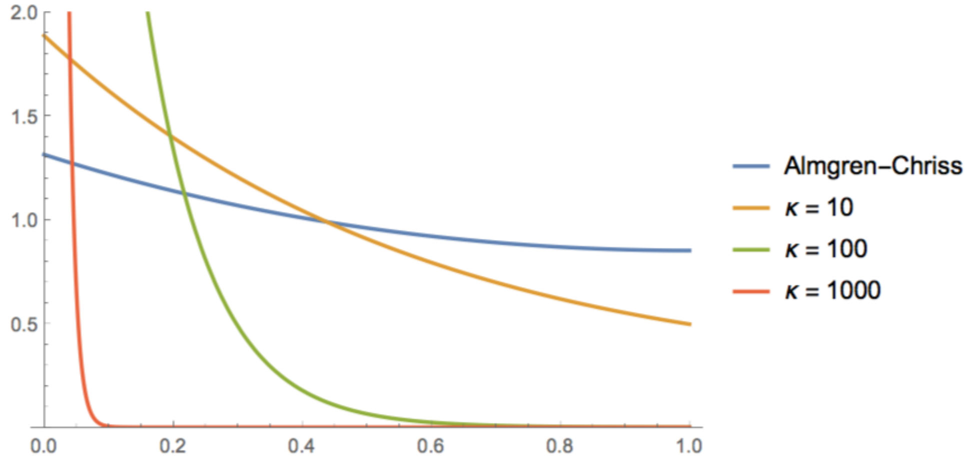

Figure 4.1.: Optimal liquidation rate ξ^* corresponding to parameters $T = 1$, $X = 1$, $\lambda = 5$ and $\eta = 5$. The dashed line corresponds to $\kappa = 0$, that is the Almgren-Chriss model with temporary impact.

For the forward component of (4.22), we have

$$X_t^* = \frac{e^{\alpha_+(T-t)} - e^{\alpha_-(T-t)}}{e^{\alpha_+ T} - e^{\alpha_- T}} X.$$

Finally, we have the optimal liquidation rate as follows,

$$\xi_t^* = \frac{\alpha_+ e^{\alpha_+(T-t)} - \alpha_- e^{\alpha_-(T-t)}}{e^{\alpha_+ T} - e^{\alpha_- T}} X. \tag{4.27}$$

When $\kappa \to 0$, $\xi_t^* \to \frac{\gamma \cosh(\gamma(T-t))}{\sinh(\gamma T)} X$ with $\gamma = \sqrt{\frac{\lambda}{\eta}}$. This corresponds to the benchmark model in [AC01]. This convergence can also be seen from Figure 1 and Figure 2. Furthermore, we see that - as in the corresponding single player models - the optimal liquidation rate is always positive, i.e. round trips are no beneficial. Moreover, we see that when the impact of interaction is strong, then the players trade very fast initially and slowly afterwards. The intuitive reason is that, when the interaction is strong, an individual player would benefit from trading fast slightly before his competitors do in order to avoid the negative drift generated by the mean-field interaction. As all the players are statistically indentical, they "coordinate" on an equilibrium trading strategy as depicted in Figure 1. Thus, our model provides a possible explanation for large price increases or decreases in markets with strategically interacting players with similar preferences.

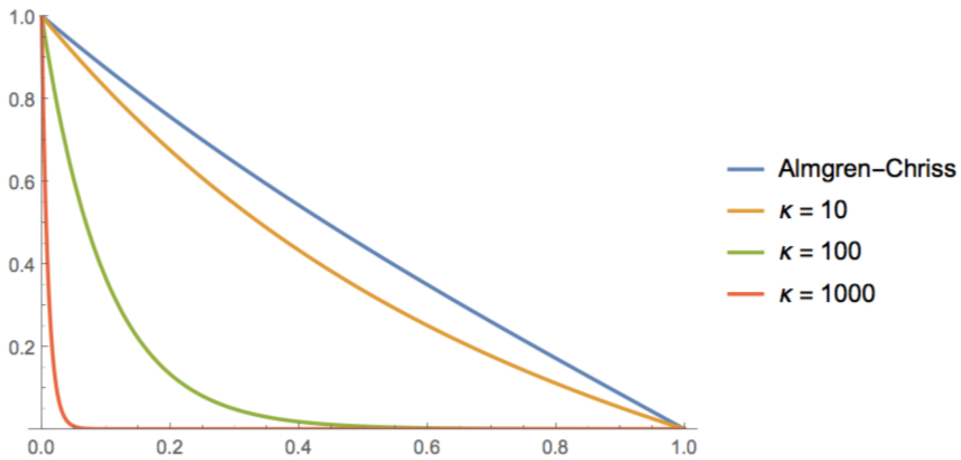

Figure 4.2.: Current state X^* corresponding to parameters $T = 1$, $X = 1$, $\lambda = 5$ and $\eta = 5$. The dashed line corresponds to $\kappa = 0$, that is the Almgren-Chriss model with temporary impact.

4.2. Penalized Optimization

In this section, we consider the penalized MFGs:

$$\begin{cases} \text{1. fix a process } \mu; \\ \text{2. solve the standard optimization problem: minimize} \\ \quad J^n(\xi;\mu) = \mathbb{E}\left[\int_0^T \left(\kappa_t\mu_t X_t + \eta_t\xi_t^2 + \lambda_t X_t^2\right)\, dt + nX_T^2\right] \\ \quad \text{such that } dX_t = -\xi_t\, dt \quad X_0 = x; \\ \text{3. solve the fixed point equation } \mu_t^* = \mathbb{E}[\xi_t^*|\mathcal{F}_t^0] \ a.e.\ t \in [0,T], \\ \quad \text{where } \xi^* \text{ is the optimal strategy from 2.} \end{cases} \tag{4.28}$$

By the same argument as in Section 4.1, the unconstrained control problem leads to the following conditional mean field FBSDE

$$\begin{cases} dX_t^n = -\left(\dfrac{A_t^n X_t^n + B_t^n}{2\eta_t}\right) dt, \\ X_0^n = x, \\ -dB_t^n = \left(-\dfrac{A_t^n B_t^n}{2\eta_t} + \kappa_t \mathbb{E}\left[\dfrac{A_t^n X_t^n + B_t^n}{2\eta_t}\middle| \mathcal{F}_t^0\right]\right) dt - Z_t^{B^n}\, d\widetilde{W}_t, \\ B_T^n = 0, \end{cases} \tag{4.29}$$

where

$$\begin{cases} -dA^n_t = \left\{2\lambda_t - \dfrac{(A^n_t)^2}{2\eta_t}\right\} dt - Z^{A^n}_t \, d\widetilde{W}_t, \\ \quad A^n_T = 2n. \end{cases} \tag{4.30}$$

The existence of a solution (A^n, Z^{A^n}) to (4.30) can be deduced from Lemma A.6.1. By Lemma A.6.2 there exists a constant $\mathfrak{C}$ such for any n, $\|A^n\|_{-1} + \|A^n\|_{n,-1} \leq \mathfrak{C}$ and A^n is a non decreasing sequence converging pointwise to A. Moreover for any $0 \leq s \leq t \leq T$ and any n, $A^n_t \leq A_t$ a.s..

Let us define the space

$$\mathcal{H}^n_\gamma = \left\{ U \in \mathcal{P}_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R}\cup\{\infty\}) : \ \left(T - . + \frac{\eta_\star}{n}\right)^{-\gamma} U. \in L^\infty_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R}\cup\{\infty\}) \right\},$$

endowed with the norm $\|\cdot\|_{n,\gamma}$

$$\|U\|_{\mathcal{H}^n_\gamma} := \|U\|_{n,\gamma} := \operatorname*{ess\,sup}_{(\omega,t)\in\Omega\times[0,T]} \left\{ \left(T - t + \frac{\eta_\star}{n}\right)^{-\gamma} |U_t| \right\}.$$

Existence and uniqueness of a solution $(X^n, B^n, Z^{B^n}) \in \mathcal{H}^n_\alpha \times \mathcal{H}^n_\alpha \times L^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R}^m)$ to (4.29) follows from similar estimates as in the proof of Theorem 4.1.5.

In order to establish the convergence of the value functions of the unconstrained penalized problems to the value function of the constrained problem we need a uniform norm estimate for the sequence (X^n, B^n). Lemma 4.2.1 provides the required estimate.

Lemma 4.2.1. *There exists a constant $\overline{\mathfrak{C}}$ and T_2 such that when $T \leq T_2$ for any n it holds that*

$$\|X^n\|_{n,\alpha} + \|B^n\|_\alpha \leq \overline{\mathfrak{C}}.$$

Proof. From (4.29) we have

$$X^n_t = x e^{-\int_0^t \frac{A^n_r}{2\eta_r} dr} - \int_0^t B^n_s e^{-\int_s^t \frac{A^n_r}{2\eta_r} dr} ds \tag{4.31}$$

and

$$B^n_t = \mathbb{E}\left[\int_t^T \kappa_s \mathbb{E}\left[\frac{A^n_s X^n_s + B^n_s}{2\eta_s} \middle| \mathcal{F}^0_s \right] e^{-\int_t^s \frac{A^n_r}{2\eta_r} dr} ds \middle| \mathcal{F}_t \right]. \tag{4.32}$$

Thus, by Appendix A.6 one has

$$\begin{aligned} |X^n_t| &\leq \frac{|x|}{T^\alpha}\left(T - t + \frac{\eta_\star}{n}\right)^\alpha + \left(T - t + \frac{\eta_\star}{n}\right)^\alpha \int_0^t \frac{|B^n_s|}{\left(T - s + \frac{\eta_\star}{n}\right)^\alpha} ds \\ &\leq \frac{|x|}{T^\alpha}\left(T - t + \frac{\eta_\star}{n}\right)^\alpha + \left(T - t + \frac{\eta_\star}{n}\right)^\alpha \int_0^t \frac{|B^n_s|}{(T-s)^\alpha} ds. \end{aligned} \tag{4.33}$$

Let us now consider the sequence B^n. With $\zeta := \|\kappa\|/(2\eta_\star)$, Lemma A.6.2 implies that

$$\frac{|B^n_t|}{(T-t)^\alpha} \leq \zeta \frac{\|A^n\|_{n,-1}}{(T-t)^\alpha} \mathbb{E}\left[\int_t^T \left(T - s + \frac{\eta_\star}{n}\right)^{-1} \mathbb{E}[|X^n_s| \, |\mathcal{F}^0_s] \, ds \middle| \mathcal{F}_t \right]$$

$$
\begin{aligned}
&+ \frac{\zeta}{(T-t)^\alpha}\mathbb{E}\left[\int_t^T \mathbb{E}[|B_s^n|\,|\mathcal{F}_s^0]\,ds\middle|\mathcal{F}_t\right] \\
&\le \frac{\zeta\mathfrak{C}|x|}{T^\beta(T-t)^\alpha}\int_t^T \left(T-s+\frac{\eta_\star}{n}\right)^{\alpha-1} ds \\
&+ \frac{\zeta\mathfrak{C}}{(T-t)^\alpha}\mathbb{E}\left[\int_t^T \left(T-s+\frac{\eta_\star}{n}\right)^{\alpha-1}\mathbb{E}\left[\int_0^s \frac{|B_u^n|}{(T-u)^\alpha}\,du\middle|\mathcal{F}_s^0\right] ds\middle|\mathcal{F}_t\right] \\
&+ \frac{\zeta}{(T-t)^\alpha}\mathbb{E}\left[\int_t^T \mathbb{E}[|B_s^n||\mathcal{F}_s^0]\,ds\middle|\mathcal{F}_t\right] \\
&\le \frac{\zeta\mathfrak{C}|x|}{\alpha T^\alpha (T-t)^\alpha}\left[\left(T-t+\frac{\eta_\star}{n}\right)^\alpha - \left(\frac{\eta_\star}{n}\right)^\alpha\right] \\
&+ \frac{\zeta\mathfrak{C}}{\alpha}\frac{1}{(T-t)^\alpha}\left[\left(T-t+\frac{\eta_\star}{n}\right)^\alpha - \left(\frac{\eta_\star}{n}\right)^\alpha\right]\int_0^T \operatorname*{ess\,sup}_{\Omega\times[0,t]}\frac{|B_s^n|}{(T-s)^\alpha}\,dt \\
&+ \zeta\int_0^T \operatorname*{ess\,sup}_{\Omega\times[0,t]}\frac{|B_s^n|}{(T-s)^\alpha}\,dt.
\end{aligned}
$$

Since the function $t \mapsto \frac{1}{(T-t)^\alpha}\left[\left(T-t+\frac{\eta_\star}{n}\right)^\alpha - \left(\frac{\eta_\star}{n}\right)^\alpha\right]$ is non negative and bounded by one,

$$
\operatorname*{ess\,sup}_{\Omega\times[0,T]}\frac{|B_t^n|}{(T-t)^\alpha} \le \frac{\zeta\mathfrak{C}|x|}{\alpha T^\alpha} + \left(\frac{\zeta\mathfrak{C}}{\alpha}+\zeta\right)\int_0^T \operatorname*{ess\,sup}_{\Omega\times[0,t]}\frac{|B_s^n|}{(T-s)^\alpha}\,dt.
$$

The conclusion can be obtained when T is small enough. $\square$

The optimality of the strategy $\xi^{n,*} = \frac{A^n X^n + B^n}{2\eta}$ can be proved using the same arguments as in the proof of Theorem 4.1.7, and in equilibrium,

$$
\mu_t^n = \mathbb{E}\left[\frac{A_t^n X_t^n + B_t^n}{2\eta_t}\middle|\mathcal{F}_t^0\right], \qquad a.e.\ t\in[0,T]. \tag{4.34}
$$

We are going to prove the convergence of the optimal strategies and portfolio processes.

Lemma 4.2.2. *There exists $0 < T_3 \le T_2$ such that when $T \le T_3$, the sequence $(A^n X^n, B^n, X^n)$ converges in $L^1([0,T]\times\Omega;\mathbb{R})$ to (AX, B, X):*

$$
\lim_{n\to+\infty}\mathbb{E}\left[\int_0^T \left(|A_t^n X_t^n - A_t X_t| + |B_t^n - B_t| + |X_t^n - X_t|\right)dt\right] = 0,
$$

Proof. From (4.31) and Lemma 4.2.1 it follows that

$$
\begin{aligned}
|X_t^n - X_t| &\le |x| \left| e^{-\int_0^t \frac{A_r^n}{2\eta_r}\,dr} - e^{-\int_0^t \frac{A_r}{2\eta_r}\,dr} \right| + \int_0^t |B_s^n - B_s| e^{-\int_s^t \frac{A_r}{2\eta_r}\,dr}\,ds \\
&\quad + \int_0^t |B_s^n| \left| e^{-\int_s^t \frac{A_r^n}{2\eta_r}\,dr} - e^{-\int_s^t \frac{A_r}{2\eta_r}\,dr} \right| ds \\
&\le \left(|x| e^{-\int_0^t \frac{A_r^n}{2\eta_r}\,dr} + \int_0^t |B_s^n|\, e^{-\int_s^t \frac{A_r^n}{2\eta_r}\,dr}\,ds \right) \left(1 - e^{-\int_0^t \frac{(A_r - A_r^n)}{2\eta_r}\,dr} \right) \\
&\quad + C(T-t)^\alpha \int_0^t \frac{|B_s^n - B_s|}{(T-s)^\alpha}\,ds \\
&\le \ell_t^n \left(\frac{T - t + \frac{\eta_\star}{n}}{T + \frac{\eta_\star}{n}} \right)^\alpha |x| + \overline{\mathfrak{C}} T \ell_t^n \left(T - t + \frac{\eta_\star}{n} \right)^\alpha \\
&\quad + C(T-t)^\alpha \int_0^t \frac{|B_s^n - B_s|}{(T-s)^\alpha}\,ds
\end{aligned}
\tag{4.35}
$$

with

$$
\ell_t^n = 1 - \exp\left(- \int_0^t \frac{(A_r - A_r^n)}{2\eta_r}\,dr \right).
$$

Notice that $0 \le \ell_t^n \le 1$ and that $\ell_t^n \overset{n\to\infty}{\longrightarrow} 1$ for each $0 \le t < T$. Furthermore, by Lemma A.6.2 and Lemma 4.1.6,

$$
\begin{aligned}
|A_t^n X_t^n - A_t X_t| &\le |A_t^n||X_t^n - X_t| + |X_t||A_t^n - A_t| \\
&\le \mathfrak{C} \left(T - t + \frac{\eta_\star}{n} \right)^{-1} |X_t^n - X_t| + \|X\|_\alpha (T-t)^\alpha |A_t^n - A_t|.
\end{aligned}
$$

We can deduce that

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T |A_t^n X_t^n - A_t X_t| dt \right] \\
&\le \mathfrak{C}\mathbb{E}\left[\int_0^T \left(T - t + \frac{\eta_\star}{n} \right)^{-1} |X_t^n - X_t| dt \right] + \|X\|_\alpha \mathbb{E}\left[\int_0^T (T-t)^\alpha |A_t^n - A_t| dt \right] \\
&\le C\mathbb{E}\left[\int_0^T \left(\frac{1}{(T + \frac{\eta_\star}{n})^\alpha} \left(T - t + \frac{\eta_\star}{n} \right)^{\alpha - 1} \ell_t^n + (T-t)^\alpha |A_t^n - A_t| \right) dt \right] \\
&\quad + C\mathbb{E}\left[\int_0^T \left(T - t + \frac{\eta_\star}{n} \right)^{\alpha-1} \ell_t^n dt \right] + C\mathbb{E}\left[\int_0^T |B_s^n - B_s| ds \right].
\end{aligned}
\tag{4.36}
$$

From (4.32) it follows that

$$
|B_t^n - B_t| \le \frac{\|\kappa\|}{2\eta_\star} \mathbb{E}\left[\int_t^T \mathbb{E}[|A_s^n X_s^n - A_s X_s||\mathcal{F}_s^0] e^{-\int_t^s \frac{A_r}{2\eta_r}\,dr} ds \,\middle|\, \mathcal{F}_t \right]
$$

$$
\begin{aligned}
&+\frac{\|\kappa\|}{2\eta_\star}\mathbb{E}\left[\int_t^T \mathbb{E}[|A_s^n X_s^n||\mathcal{F}_s^0] e^{-\int_t^s \frac{A_r^n}{2\eta_r}\,dr}\left(1-e^{-\int_t^s \frac{(A_r-A_r^n)}{2\eta_r}\,dr}\right) ds\middle|\mathcal{F}_t\right] \\
&+\frac{\|\kappa\|}{2\eta_\star}\mathbb{E}\left[\int_t^T \mathbb{E}[|B_s^n - B_s||\mathcal{F}_s^0]\,ds\middle|\mathcal{F}_t\right] \\
&+\|B\|_\alpha T^\alpha \mathbb{E}\left[\int_t^T \left|e^{-\int_t^s \frac{A_r^n}{2\eta_r}\,dr} - e^{-\int_t^s \frac{A_r}{2\eta_r}\,dr}\right| ds\right] \\
\leq{}& \frac{\|\kappa\|}{2\eta_\star}\mathbb{E}\left[\int_t^T \mathbb{E}[|A_s^n X_s^n - A_s X_s||\mathcal{F}_s^0]\,ds\middle|\mathcal{F}_t\right] \\
&+\frac{\|\kappa\|\overline{\mathfrak{C}}\mathfrak{C}}{2\eta_\star}\mathbb{E}\left[\int_t^T \left(T-s+\frac{\eta_\star}{n}\right)^{\beta-1} e^{-\int_t^s \frac{A_r^n}{2\eta_r}\,dr}\ell_s^n\,ds\middle|\mathcal{F}_t\right] \\
&+\frac{\|\kappa\|}{2\eta_\star}\mathbb{E}\left[\int_t^T \mathbb{E}[|B_s^n - B_s||\mathcal{F}_s^0]\,ds\middle|\mathcal{F}_t\right] + \|B\|_\alpha T^\alpha \mathbb{E}\left[\int_0^T \ell_s^n\,ds\right].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathbb{E}\left[\int_0^T |B_t^n - B_t|dt\right] \leq{}& \frac{\|\kappa\|}{2\eta_\star}\mathbb{E}\left[\int_0^T\left(\int_t^T |A_s^n X_s^n - A_s X_s|\,ds\right)dt\right] \\
&+\frac{\|\kappa\|\overline{\mathfrak{C}}\mathfrak{C}}{2\eta_\star}\mathbb{E}\left[\int_0^T\left(\int_t^T \left(T-s+\frac{\eta_\star}{n}\right)^{\alpha-1} e^{-\int_t^s \frac{A_r^n}{2\eta_r}\,dr}\ell_s^n\,ds\right)dt\right] \\
&+\frac{\|\kappa\|}{2\eta_\star}\mathbb{E}\left[\int_0^T\left(\int_t^T |B_s^n - B_s|\,ds\right)dt\right] + \|B\|_\alpha T^{\alpha+1}\mathbb{E}\left[\int_0^T \ell_s^n\,ds\right].
\end{aligned}
$$

Since $A^n \in \mathcal{H}_{-1}^n$, $A \in \mathcal{H}_{-1}$, $X^n \in \mathcal{H}_\alpha^n$, $X \in \mathcal{H}_\alpha$, $B^n \in \mathcal{H}_\alpha$ and $B \in \mathcal{H}_\alpha$, all the above integrals are well defined. Plugging this back to (4.36), when T is small enough, one has

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T |B_t^n - B_t|dt\right] + \mathbb{E}\left[\int_0^T |A_t^n X_t^n - A_t X_t|dt\right] \\
\leq{}& C\mathbb{E}\left[\int_0^T\left(\frac{1}{(T+\frac{\eta_\star}{n})^\alpha}\left(T-t+\frac{\eta_\star}{n}\right)^{\alpha-1}\ell_t^n + (T-t)^\alpha|A_t^n - A_t|\right)dt\right] \\
&+C\mathbb{E}\left[\int_0^T\left(T-t+\frac{1}{2n}\right)^{\alpha-1}\ell_t^n dt\right] + C\mathbb{E}\left[\int_0^T \ell_s^n\,ds\right] \\
&+C\mathbb{E}\left[\int_0^T\left(\int_t^T\left(T-s+\frac{\eta_\star}{n}\right)^{\alpha-1} e^{-\int_t^s \frac{A_r^n}{2\eta_r}\,dr}\ell_s^n\,ds\right)dt\right].
\end{aligned}
$$

In view of the following five estimates

- $$0 \leq \ell_s^n \leq 1,$$

- $$\frac{1}{\left(t+\frac{\eta_\star}{n}\right)^\alpha}\int_0^t \left(t-s+\frac{\eta_\star}{n}\right)^{\alpha-1}\ell_s^n ds \le 1,$$

- $$\left(t-s+\frac{\eta_\star}{n}\right)^{\alpha-1}\ell_s^n \le (t-s)^{\alpha-1},$$

  which is integrable in s,

- $$(T-t)^\alpha |A_t^n - A_t| \le (T-t)^{\alpha-1}(\|A^n\|_{-1}+\|A\|_{-1}) \le 2\mathfrak{C}(T-t)^{\alpha-1},$$

  which is integrable in t,

- $$\left(t-u+\frac{\eta_\star}{n}\right)^{\alpha-1} e^{-\int_s^u \frac{A_r^n}{2\eta_r}dr}\ell_u^n \le (t-u)^{\alpha-1},$$

  which is integrable in u,

dominated convergence yield that

$$\lim_{n\to+\infty}\mathbb{E}\left[\int_0^T |A_t^n X_t^n - A_t X_t| dt\right] = 0.$$

and

$$\lim_{n\to+\infty}\mathbb{E}\left[\int_0^T |B_t^n - B_t| dt\right] = 0.$$

Hence, it follows from (4.35) and Fubini's theorem that

$$\begin{aligned}
& \mathbb{E}\left[\int_0^T |X_t^n - X_t| dt\right] \\
\le\ & C\mathbb{E}\left[\int_0^T \ell_t^n \left(T-t+\frac{1}{2n}\right)^\alpha dt\right] + C\mathbb{E}\left[\int_0^T \ell_t^n \left(T-t+\frac{1}{2n}\right)^\alpha dt\right] \\
& + C\mathbb{E}\left[\int_0^T |B_s^n - B_s|\, ds\right].
\end{aligned}$$

This yields the desired result. □

Let us denote by $V^n(t,x;\mu^n)$ the value function associated with the penalized problem (4.28). The next theorem shows the convergence of $V^n(0,x;\mu^n) := V^n(x)$ to the value function $V(0,x;\mu) := V(x)$ associated with the contarined MFG.

Theorem 4.2.3. *The value function $V^n(x)$ converges to $V(x)$.*

Proof. Any admissible control ξ of the original problem is admissible for this penalized setting. Hence we have immediately that $V^n \le V$.

For any $\varepsilon > 0$

$$\begin{aligned}
V(x) &= \mathbb{E}\left[\int_0^T \left(\kappa_s \mathbb{E}\left[\frac{A_s X_s + B_s}{2\eta_s}\middle|\mathcal{F}_s^0\right] X_s + \eta_s \xi_s^2 + \lambda_s X_s^2\right) ds\right] \\
&\ge \mathbb{E}\left[\int_0^T \left(\kappa_s \mathbb{E}\left[\frac{A_s^n X_s^n + B_s^n}{2\eta_s}\middle|\mathcal{F}_s^0\right] X_s^n + \eta_s (\xi_s^n)^2 + \lambda_s (X_s^n)^2\right) ds + n(X_T^n)^2\right] \\
&= V^n(x) \\
&\ge \mathbb{E}\left[\int_0^T \left(\kappa_s \mathbb{E}\left[\frac{A_s^n X_s^n + B_s^n}{2\eta_s}\middle|\mathcal{F}_s^0\right] X_s^n + \eta_s (\xi_s^n)^2 + \lambda_s (X_s^n)^2\right) ds\right] \\
&\ge \mathbb{E}\left[\int_0^T \left(\kappa_s \mathbb{E}\left[\frac{A_s^n X_s^n + B_s^n}{2\eta_s}\middle|\mathcal{F}_s^0\right] X_s^n\right) ds\right] \\
&\quad + \mathbb{E}\left[\int_0^{T-\varepsilon} \left(\eta_s (\xi_s^n)^2 + \lambda_s (X_s^n)^2\right) ds\right].
\end{aligned} \tag{4.37}$$

Recall that the first inequality comes from the fact that $\xi_s = (A_s X_s + B_s)/(2\eta_s)$ is also an admissible control for the penalized problem.

Let us first consider

$$\mathcal{V}^{n,1} := \mathbb{E}\left[\int_0^{T-\varepsilon} \left(\eta_s (\xi_s^n)^2 + \lambda_s (X_s^n)^2\right) ds\right] - \mathbb{E}\left[\int_0^{T-\varepsilon} \left(\eta_s (\xi_s)^2 + \lambda_s (X_s)^2\right) ds\right].$$

We can split $\mathcal{V}^{n,1}$ as follows:

$$\begin{aligned}
|\mathcal{V}^{n,1}| \le & \frac{1}{4\eta_\star}\mathbb{E}\left[\int_0^{T-\varepsilon} |B_t^n - B_t||A_t^n X_t^n + B_t^n + A_t X_t + B_t|\, dt\right] \\
& + \frac{1}{4\eta_\star}\mathbb{E}\left[\int_0^{T-\varepsilon} |A_t^n X_t^n - A_t X_t||A_t^n X_t^n + B_t^n + A_t X_t + B_t|\, dt\right] \\
& + \|\lambda\|\mathbb{E}\left[\int_0^{T-\varepsilon} |X_t^n - X_t||X_t^n + X_t|\, dt\right].
\end{aligned}$$

On the other hand, from Lemma 4.2.1, following estimates hold uniformly in n

$$\operatorname*{ess\,sup}_{\Omega\times[0,T]} \frac{|X_t^n|}{\left(T - t + \frac{\eta_\star}{n}\right)^\alpha} < \infty, \qquad \operatorname*{ess\,sup}_{\Omega\times[0,T]} \frac{|B_t^n|}{(T-t)^\alpha} < \infty.$$

The first estimate and Lemma A.6.2 yield that uniformly in n

$$\operatorname*{ess\,sup}_{\Omega\times[0,T]} \frac{|A_t^n X_t^n|}{\left(T - t + \frac{\eta_\star}{n}\right)^{\alpha-1}} < \infty,$$

which means that A^nX^n is uniformy (w.r.t. n) bounded in $L^\infty([0,T-\varepsilon]\times\Omega;\mathbb{R})$. Moreover, from Lemma 4.2.2, there exists a common subsequence such that $\mathbb{P}\otimes dt$ a.e.

$$A^n_tX^n_t \to A_tX_t, \qquad B^n_t \to B_t, \qquad X^n_t \to X_t.$$

Hence, by dominated convergence,

$$\lim_{n\to+\infty}\mathbb{E}\left[\int_0^{T-\varepsilon}\left(\eta_s(\xi^n_s)^2+\lambda_s(X^n_s)^2\right)\,ds\right]=\mathbb{E}\left[\int_0^{T-\varepsilon}\left(\eta_s(\xi_s)^2+\lambda_s(X_s)^2\right)\,ds\right].$$

Now if we define

$$\begin{aligned}\mathcal{V}^{n,2}:=\;&\mathbb{E}\left[\int_0^T\left(\kappa_s\mathbb{E}\left[\frac{A^n_sX^n_s+B^n_s}{2\eta_s}\middle|\mathcal{F}^0_s\right]X^n_s\right)\,ds\right]\\&-\mathbb{E}\left[\int_0^T\left(\kappa_s\mathbb{E}\left[\frac{A_sX_s+B_s}{2\eta_s}\middle|\mathcal{F}^0_s\right]X_s\right)\,ds\right],\end{aligned}$$

then we have that

$$\begin{aligned}|\mathcal{V}^{n,2}|\le\;&\frac{\|\kappa\|}{2\eta_\star}\mathbb{E}\left[\int_0^T\mathbb{E}[|A^n_tX^n_t+B^n_t-A_tX_t-B_t||\mathcal{F}^0_t]|X^n_t|\,dt\right]\\&+\frac{\|\kappa\|}{2\eta_\star}\mathbb{E}\left[\int_0^T\mathbb{E}[|A_tX_t+B_t||\mathcal{F}^0_t]|X^n_t-X_t|\,dt\right].\end{aligned}$$

From Lemma 4.1.6 and Lemma 4.2.1, a.s. for any t and any n

$$\begin{aligned}&\mathbb{E}[|A_tX_t+B_t||\mathcal{F}^0_t]|X^n_t-X_t|\\&\le C(\sup_n\|X^n\|_{n,\alpha}+\|X\|_\alpha)\mathbb{E}[|A_tX_t+B_t||\mathcal{F}^0_t]\in L^2_{\mathbb{F}^0}([0,T]\times\Omega;\mathbb{R})\end{aligned}$$

and

$$\mathbb{E}[|A^n_tX^n_t+B^n_t-A_tX_t-B_t||\mathcal{F}^0_t]|X^n_t|\le C\sup_n\|X^n\|_{n,\alpha}\mathbb{E}[|A^n_tX^n_t+B^n_t-A_tX_t-B_t||\mathcal{F}^0_t].$$

By the Vitali convergence theorem and Lemma 4.2.2, we get that $\mathcal{V}^{n,2}$ tends to zero when n goes to $+\infty$.

By (4.37),

$$\begin{aligned}&V(x)\ge V^n(x)\\&\ge\mathbb{E}\left[\int_0^T\left(\kappa_s\mathbb{E}\left[\frac{A^n_sX^n_s+B^n_s}{2\eta_s}\middle|\mathcal{F}^0_s\right]X^n_s\right)\,ds\right]+\mathbb{E}\left[\int_0^{T-\varepsilon}\left(\eta_s(\xi^n_s)^2+\lambda_s(X^n_s)^2\right)\,ds\right]\end{aligned}$$

and the right-hand side converges to

$$\mathbb{E}\left[\int_0^T\left(\kappa_s\mathbb{E}\left[\frac{A_sX_s+B_s}{2\eta_s}\middle|\mathcal{F}^0_s\right]X_s\right)\,ds\right]+\mathbb{E}\left[\int_0^{T-\varepsilon}\left(\eta_s(\xi_s)^2+\lambda_s(X_s)^2\right)\,ds\right]$$

when n goes to $+\infty$. The monotone convergence theorem applied to ε gives the desired result. $\square$

Remark 4.2.4. As a by-product of the proof, we get that $\lim_{n\to+\infty} \mathbb{E}\left[n(X_T^n)^2\right] = 0$. Moreover, we have as well

$$|X_T^n| \leq \left(\frac{x}{T^\beta} + T\|B^n\|_\eta\right)\frac{1}{n^\beta} \leq \left(\frac{x}{T^\beta} + T\overline{\mathfrak{C}}\right)\frac{1}{n^\beta}.$$

Remark 4.2.5. The proof of convergence of the value function simplifies substantially in the setting of Section 4.1.3. Moreover, we could have global convergence in the common value environment. In this case, $Y^n = A^n X^n$ where

$$-dA_t^n = \left(2\lambda_t + \frac{\kappa_t A_t^n}{2\eta_t} - \frac{(A_t^n)^2}{2\eta_t}\right) dt - Z_t^{A^n} dW_t^0, \quad A_T^n = 2n$$

and

$$dX_t^n = -\frac{A_t^n X_t^n}{2\eta_t}\, dt, \;\; X_0 = x.$$

The optimal strategy and the resulting portfolio process are given by, respectively,

$$\xi_t^{n,*} = \mu_t^{n,*} = \frac{A_t^n X_t^{n,*}}{2\eta_t}, \quad X_t^{n,*} = xe^{-\int_0^t \frac{A_r^n}{2\eta_r}\, dr} \qquad t \in [0,T].$$

Since the sequence A^n is non decreasing and converges to A, we deduce that $X^{n,*}$ converges to X^* a.e. a.s. and that $\xi^{n,*}$ converges to ξ^* a.e. a.s.. For any any admissible strategy $\xi \in \mathcal{A}_{\mathbb{F}^0}(t,x)$ with associated portfolio process X,

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T \left(\kappa_s\xi_s X_s + \eta_s\xi_s^2 + \lambda_s X_s^2\right) ds \,\middle|\, \mathcal{F}_t^0\right] \\
&\geq \mathbb{E}\left[\int_t^T \left(\kappa_s\xi_s^{n,*}X_s^{n,*} + \eta_s(\xi_s^{n,*})^2 + \lambda_s(X_s^{n,*})^2\right) ds + n(X_T^{n,*})^2 \,\middle|\, \mathcal{F}_t^0\right] \\
&\geq \mathbb{E}\left[\int_t^T \left(\kappa_s\xi_s^{n,*}X_s^{n,*} + \eta_s(\xi_s^{n,*})^2 + \lambda_s(X_s^{n,*})^2\right) ds \,\middle|\, \mathcal{F}_t^0\right].
\end{aligned}$$

For any $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to+\infty} \mathbb{E}\left[\int_t^{T-\varepsilon} \left(\kappa_s\xi_s^{n,*}X_s^{n,*} + \eta_s(\xi_s^{n,*})^2 + \lambda_s(X_s^{n,*})^2\right) ds \,\middle|\, \mathcal{F}_t^0\right] \\
&= \mathbb{E}\left[\int_t^{T-\varepsilon} \left(\kappa_s\xi_s^* X_s + \eta_s(\xi_s^*)^2 + \lambda_s(X_s^*)^2\right) ds \,\middle|\, \mathcal{F}_t^0\right].
\end{aligned}$$

Hence, the monotone convergence theorem gives the desired convergence.

5. PART II-3: Leader Follower Mean Field Games with Terminal Constraint

In this chapter, we consider a leader-follower MFGs with constraint, arising from optimal portfolio liquidation of two groups of players. In contrast to the MFG liquidation model studied in Chapter 4, we assume there are one leader and N followers. The quantity with index 0 is for the leader and the one with i is for follower i. For example, the trading rates of the leader and follower i are ξ^0 and ξ^i, $i = 1, 2, \cdots, N$, resp. The current positions are X^0 and X^i, $i = 1, 2, \cdots, N$, resp. Similarly as Chapter 4, the trading prices for leader and the ith follower are of the form, respectively

$$S_t^0 = S_0 - \int_0^t \frac{\kappa}{N} \sum_{j=1}^{N} \xi_s^j \, ds - \int_0^t \kappa^0 \xi_s^0 \, ds + \sigma W_t^0 - \eta^0 \xi_t^0. \tag{5.1}$$

and

$$S_t^i = S_0 - \int_0^t \frac{\kappa}{N} \sum_{j=1}^{N} \xi_s^j \, ds - \int_0^t \kappa^0 \xi_s^0 \, ds + \sigma W_t^0 - \eta \xi_t^i. \tag{5.2}$$

Thus, the optimization problems of leader and followers are summarized as follows:

- *Leader's Problem:* minimize

$$\mathbb{E}\left[\int_0^T \left(\frac{\kappa}{N} \sum_{j=1}^{N} \xi_t^j X_t^0 + \kappa^0 X_t^0 \xi_t^0 + \eta^0 (\xi_t^0)^2 + \lambda^0 (X_t^0)^2 + \overline{\lambda}_t \left(\frac{1}{N} \sum_{j=1}^{N} \xi_t^j \right)^2 \right) dt \right]^{1} \tag{5.3}$$

  subject to

$$dX_t^0 = -\xi_t^0 \, dt, \qquad X_0 = x^0 \text{ and } X_T^0 = 0;$$

- *Generic follower's Problem:* minimize

$$\mathbb{E}\left[\int_0^T \left(\frac{\kappa_t X_t^i}{N} \sum_{j}^{N} \xi_t^j + \kappa_t^0 \xi_t^0 X_t^i + \eta_t^i (\xi_t^i)^2 + \lambda_t^i (X_t^i)^2 \right) dt \right] \tag{5.4}$$

  subject to

$$dX_t^i = -\xi_t^i \, dt, \qquad X_0^i = x \text{ and } X_T^i = 0.$$

[1] The last additional terminal is to regularize the optimization problem for the leader.

Similarly to the solvability of the classical leader-follower game (see, e.g. [Yon02]), the idea is: for each choice ξ^0 of the leader, the follower would like to choose ξ to minimize her cost; by knowing the follower would take the optimal strategy ξ^* which depends on ξ^0, the leader would take a corresponding strategy to minimize her cost. Mathematically, it is split into two steps

Step 1. The solvability of the representative follower's problem.

$$\begin{cases} \text{1. Fix a strategy of the leader } \xi^0 \text{ and a mean filed } \mu; \\ \text{2. Solve the optimization problem :} \\ \quad \inf_{\xi} \mathbb{E}\left[\int_0^T \left(\kappa_t \mu_t X_t + \kappa_t^0 \xi_t^0 X_t + \eta_t \xi_t^2 + \lambda_t X_t^2\right) dt\right] \\ \quad \text{subject to } dX_t = -\xi_t \, dt, \quad X_0 = x, \quad X_T = 0; \\ \text{3. Search for the fixed point: } \mu_t = \mathbb{E}[\xi_t^* | \mathcal{F}_t^0], a.s.\ a.e., \\ \quad\ \text{where } \xi^* \text{ is the optimal control from 2.} \end{cases}$$

Step 2. The solvability of the leader's problem.
Let $\mu^*(\xi^0)$ be the optimal strategy and the equilibrium for the representative follower's problem, respectively. Thus, the leader's problem is to minimize

$$\mathbb{E}\left[\int_0^T \left(\kappa_t \mu_t^*(\xi^0) X_t^0 + \kappa_t^0 X_t^0 \xi_t^0 + \eta_t^0 (\xi_t^0)^2 + \lambda_t^0 (X_t^0)^2 + \overline{\lambda}_t (\mu_t^*(\xi^0))^2\right) dt\right], \tag{5.5}$$

subject to

$$\begin{cases} dX_t^0 = -\xi_t^0 \, dt, \quad X_T^0 = 0, \\ \text{and the dynamics of } \mu^*(\xi^0). \end{cases}$$

This leads to a stochastic control problem for McKean-Vlasov FBSDE with state constraint.

By analogy to the model studied in Chapter 4, we assume the private information pattern. Hence, throughout this chapter, we use the following notation:

Notation. Let W^0 and W be standard Browian motions which are independent of each other. Set $\mathcal{F}_t^0 = \sigma(W_s^0, 0 \leq s \leq t)$ and $\mathcal{F}_t = \sigma(W_s, W_s^0, 0 \leq s \leq t)$. Denote by $\mathbb{F}^0 = (\mathcal{F}_t^0)_{0 \leq t \leq T}$ and $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$. Admissible control spaces for the leader and the follower are assumed to be, respectively

$$\mathcal{A}_{\mathbb{F}^0}(t, x^0) := \left\{\xi^0 \in L^2_{\mathbb{F}^0}([0, T] \times \Omega; \mathbb{R}) : \int_t^T \xi_s^0 \, ds = x^0\right\}$$

and

$$\mathcal{A}_{\mathbb{F}}(t, x) := \left\{\xi \in L^2_{\mathbb{F}}([0, T] \times \Omega; \mathbb{R}) : \int_t^T \xi_s \, ds = x\right\}.$$

The same as Chapter 4, let $\alpha := \underline{\eta}/\overline{\eta}$ and let β be any fixed constant in $(0,1/2)$ the implication of which will be clear later. Moreover, we use the convention that C is a generic positive constant which may vary from line to line.

Assumption 5.0.1. (1) κ, η, $1/\eta$ and λ belong to $L^\infty_{\mathbb{F}}([0,T]\times\Omega;[0,\infty);\mathbb{R})$.

(2) κ^0, η^0, $1/\eta^0$ and λ^0 belong to $L^\infty_{\mathbb{F}^0}([0,T]\times\Omega;[0,\infty);\mathbb{R})$.

(3) $\alpha > 1/2$.

The follower's and leader's problems are solved, respectively, in Section 5.1 and Section 5.2. In fact, the follower's problem is an MFG with constraint while the leader's problem is a stochastic control problem of McKean-Vlasov type with constraint. Both problems reduce to the solvability of singular FBSDEs of McKean-Vlasov type.

5.1. Follower's Problem–Solvability of Singular FBSDE

This section deals with the follower's problem, that is the MFG with contraint. For each fixed $\mu \in L^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R})$ and each fixed strategy of the leader $\xi^0 \in L^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R})$ which is considered to be exogenous to the MFG, define the Hamiltonian as

$$H(\xi, X, Y; \mu, \xi^0) = -\xi Y + \kappa\mu X + \kappa^0 X\xi^0 + \eta\xi^2 + \lambda X^2. \tag{5.6}$$

By analogy to Chapter 4 we have the following candidate for the optimal control

$$\xi = \frac{AX+B}{2\eta}, \tag{5.7}$$

and the maximum principle yields the following FBSDE

$$\begin{cases} dX_t &= -\dfrac{A_tX_t+B_t}{2\eta_t}, \\ -dY_t &= \left(\kappa_t\mu_t + \kappa^0_t\xi^0_t + 2\lambda_tX_t\right)\,dt - Z_t\,d\overline{W}_t, \\ X_0 &= x, \quad X_T = 0. \end{cases} \tag{5.8}$$

Let $Y = AX + B$. The probabilistic method of MFGs yields that

$$\begin{cases} dX_t = -\left(\dfrac{A_t}{2\eta}X_t + \dfrac{B_t}{2\eta_t}\right)\,dt \\ -dB_t = \left(-\dfrac{A_tB_t}{2\eta_t} + \kappa_t\mathbb{E}\left[\dfrac{A_tX_t+B_t}{2\eta_t}\middle|\mathcal{F}^0_t\right] + \kappa^0_t\xi^0_t\right)\,dt - Z_t\,d\overline{W}_t, \\ X_0 = x,\ B_T = 0, \end{cases} \tag{5.9}$$

with $\overline{W} = (W, W^0)$ and

$$-dA_t = \left(2\lambda_t - \frac{A_t^2}{2\eta_t}\right)\,dt - Z^A_t\,d\overline{W}_t,\ A_T = \infty.$$

Note that we only have $\xi^0 \in L^2_{\mathbb{F}^0}(\Omega \times [0,T];\mathbb{R})$, which implies weaker regularity for the coefficients of (5.9). Thus, in order to solve (5.9), we introduce the following space which is different from the uniform space in Chapter 4:

$$\overline{\mathcal{H}}_\epsilon = \left\{ S \in L^0_{\mathbb{F}}([0,T] \times \Omega, \overline{\mathcal{F}}) : \mathbb{E}\left[\sup_{t\in[0,T]} \left| \frac{|S_t|}{(T-t)^\epsilon} \right|^2 \right] < \infty \right\},$$

with the norm

$$\|S\|_{\overline{\mathcal{H}}_\epsilon} := \|S\|_{\overline{\epsilon}} := \left(\mathbb{E}\left[\sup_{t\in[0,T]} \left| \frac{|S_t|}{(T-t)^\epsilon} \right|^2 \right] \right)^{\frac{1}{2}}.$$

By analogy to Chapter 4, we first give an estimate for the martingale part.

Lemma 5.1.1. *Suppose Assumption 5.0.1(1-2) hold and there exists a solution to* (5.9) *such that* $(X,B) \in \overline{\mathcal{H}}_\alpha \times \overline{\mathcal{H}}_\beta$. *Then we have the following estimate*

$$\begin{aligned} &\mathbb{E}\left[\int_0^T Z_t^2\, dt\right] \\ &\le C\left(\mathbb{E}\left[\int_0^T |\xi^0_s|^2\, ds\right] + \mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{B_t}{(T-t)^\beta}\right|^2\right] + \mathbb{E}\left[\sup_{0\le s\le T}\left|\frac{X_s}{(T-s)^\alpha}\right|^2\right]\right) \end{aligned} \tag{5.10}$$

Proof. From (5.9), we have

$$\begin{aligned} &\mathbb{E}\left[\left|\int_0^T Z_t\, d\overline{W}_t\right|^2\right] \\ &\le \mathbb{E}[|B_0|^2] + \mathbb{E}[|B_T|^2] + \|\kappa^0\| T \mathbb{E}\left[\int_0^T |\xi^0_s|^2\, ds\right] \\ &\quad + \frac{1}{2\underline{\eta}}\mathbb{E}\left[\left|\int_0^T A_s B_s\, ds\right|^2\right] + \|\kappa\|\mathbb{E}\left[\left|\int_0^T \mathbb{E}\left[\left.\frac{A_s X_s + B_s}{2\eta_s}\right| \mathcal{F}^0_s\right]\, ds\right|^2\right] \qquad (5.11) \\ &\le 2T^{2\beta}\mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{B_t}{(T-t)^\beta}\right|^2\right] + \|\kappa^0\| T \mathbb{E}\left[\int_0^T |\xi^0_s|^2\, ds\right] \\ &\quad + \frac{1}{2\underline{\eta}}\mathbb{E}\left[\left|\int_0^T A_s B_s\, ds\right|^2\right] + \frac{\|\kappa\|}{2\underline{\eta}}\mathbb{E}\left[\left|\int_0^T \mathbb{E}\left[A_s X_s + B_s \middle| \mathcal{F}^0_s\right]\, ds\right|^2\right]. \end{aligned}$$

It remains to estimate the last two terms in (5.11). Firstly,

$$
\begin{aligned}
&\mathbb{E}\left[\left|\int_0^T A_s B_s\,ds\right|^2\right] \\
&\le C\mathbb{E}\left[\left|\int_0^T \frac{B_s}{T-s}\,ds\right|^2\right] \\
&\le C\mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{B_t}{(T-t)^\beta}\right|^2\left|\int_0^T \frac{1}{(T-s)^{1-\beta}}\,ds\right|^2\right] \\
&\le C\mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{B_t}{(T-t)^\beta}\right|^2\right].
\end{aligned}
\tag{5.12}
$$

Secondly,

$$
\begin{aligned}
&\mathbb{E}\left[\left|\int_0^T \mathbb{E}\left[A_s X_s\middle|\mathcal{F}_s^0\right]ds\right|^2\right] \le C\mathbb{E}\left[\left|\int_0^T \mathbb{E}\left[\frac{X_s}{T-s}\middle|\mathcal{F}_s^0\right]ds\right|^2\right] \\
&\le C\mathbb{E}\left[\sup_{0\le t\le T}\left|\mathbb{E}\left[\frac{X_t}{(T-t)^\alpha}\middle|\mathcal{F}_t^0\right]\right|^2\left|\int_0^T \frac{1}{(T-s)^{1-\alpha}}\,ds\right|^2\right] \\
&\le C\mathbb{E}\left[\sup_{0\le t\le T}\left|\mathbb{E}\left[\frac{X_t}{(T-t)^\alpha}\middle|\mathcal{F}_t^0\right]\right|^2\right] \\
&\le C\mathbb{E}\left[\sup_{0\le t\le T}\left|\mathbb{E}\left[\sup_{0\le s\le T}\frac{X_s}{(T-s)^\alpha}\middle|\mathcal{F}_t^0\right]\right|^2\right] \\
&\le C\mathbb{E}\left[\sup_{0\le s\le T}\left|\frac{X_s}{(T-s)^\alpha}\right|^2\right] \qquad \text{(Doob's maximal inequality).}
\end{aligned}
\tag{5.13}
$$

Thus, (5.11), (5.12) and (5.13) yield (5.10). □

The following theorem presents the existence and uniqueness result.

Theorem 5.1.2. *There exists $T_1 > 0$ such that for each given $\xi^0 \in L^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R})$, (5.9) admits a unique solution $(X,B,Z) \in \overline{\mathcal{H}}_\alpha \times \overline{\mathcal{H}}_\beta \times L^2_{\mathbb{F}}(\Omega\times[0,T];\mathbb{R}^m)$ if $T < T_1$.*

Proof. For each $(x,b,z) \in \overline{\mathcal{H}}_\alpha \times \overline{\mathcal{H}}_\beta \times L^2_{\mathbb{F}}(\Omega\times[0,T];\mathbb{R}^m)$, we introduce the following equation

$$
\begin{cases}
dX_t = -\left(\dfrac{A_t}{2\eta}X_t + \dfrac{b_t}{2\eta_t}\right)dt \\
-dB_t = \left(-\dfrac{A_tB_t}{2\eta_t} + \kappa_t\mathbb{E}\left[\dfrac{A_tX_t+b_t}{2\eta_t}\middle|\mathcal{F}_t^0\right] + \kappa_t^0\xi_t^0\right)dt - Z_t\,d\overline{W}_t, \\
X_0 = x,\ B_T = 0,
\end{cases}
\tag{5.14}
$$

which implies that

$$X_t = xe^{-\int_0^t \frac{A_r}{2\eta_r}\,dr} - \int_0^t \frac{b_s}{2\eta_s} e^{-\int_s^t \frac{A_r}{2\eta_r}\,dr}\,ds \tag{5.15}$$

and

$$B_t = \mathbb{E}\left[\int_t^T \left(\kappa_s \mathbb{E}\left[\frac{A_s X_s + b_s}{2\eta_s}\middle|\mathcal{F}_s^0\right] + \kappa_s^0 \xi_s^0\right) e^{-\int_t^s \frac{A_r}{2\eta_r}\,dr}\,ds\middle|\mathcal{F}_t\right]. \tag{5.16}$$

From (5.15), we have

$$\sup_{0\le t\le T}\frac{|X_t|}{(T-t)^\alpha} \le \frac{x}{T^\alpha} + C\sup_{0\le t\le T}\frac{|b_t|}{(T-t)^\beta}.$$

From (5.16), we have

$$\begin{aligned}
\frac{|B_t|}{(T-t)^\beta} &\le \frac{C}{(T-t)^\beta}\mathbb{E}\left[\int_t^T E\left[\frac{|X_s|}{T-s}\middle|\mathcal{F}_s^0\right]\,ds\middle|\mathcal{F}_t\right] \\
&\quad + C\mathbb{E}\left[\int_t^T \mathbb{E}\left[\frac{|b_s|}{(T-s)^\beta}\middle|\mathcal{F}_s^0\right]\,ds\middle|\mathcal{F}_t\right] \\
&\quad + C\mathbb{E}\left[\int_t^T |\xi_s^0|^{\frac{1}{1-\beta}}\,ds\middle|\mathcal{F}_t\right]^{1-\beta} \\
&\le \frac{C}{(T-t)^\beta}\mathbb{E}\left[\int_t^T (T-s)^{\alpha-1}\mathbb{E}\left[\int_0^s \frac{|b_r|}{(T-r)^\alpha}\,dr\middle|\mathcal{F}_s^0\right]\,ds\middle|\mathcal{F}_t\right] \\
&\quad + C\mathbb{E}\left[\int_t^T \mathbb{E}\left[\frac{|b_s|}{(T-s)^\beta}\middle|\mathcal{F}_s^0\right]\,ds\middle|\mathcal{F}_t\right] \\
&\quad + C\mathbb{E}\left[\int_t^T |\xi_s^0|^{\frac{1}{1-\beta}}\,ds\middle|\mathcal{F}_t\right]^{1-\beta} + C.
\end{aligned} \tag{5.17}$$

For the fist term above which is denoted by I_1, Hölder inequality yields that

$$\begin{aligned}
I_1 &\le C\left|\mathbb{E}\left[\int_t^T \left|(T-s)^{\alpha-1}\mathbb{E}\left[\int_0^s \frac{|b_r|}{(T-r)^\alpha}\,dr\middle|\mathcal{F}_s^0\right]\right|^{\frac{1}{1-\beta}}\,ds\middle|\mathcal{F}_t\right]\right|^{1-\beta} \\
&\le C\left|\mathbb{E}\left[\int_0^T (T-s)^{\frac{\alpha-1}{1-\beta}}\left|\mathbb{E}\left[\sup_{0\le r\le T}\frac{|b_r|}{(T-r)^\beta}\int_0^T \frac{1}{(T-r)^{\alpha-\beta}}\,dr\middle|\mathcal{F}_s^0\right]\right|^{\frac{1}{1-\beta}}\,ds\middle|\mathcal{F}_t\right]\right|^{1-\beta} \\
&\le C\left|\mathbb{E}\left[\int_0^T (T-s)^{\frac{\alpha-1}{1-\beta}}\,ds\,\sup_{0\le s\le T}\left|\mathbb{E}\left[\sup_{0\le r\le T}\frac{|b_r|}{(T-r)^\beta}\middle|\mathcal{F}_s^0\right]\right|^{\frac{1}{1-\beta}}\middle|\mathcal{F}_t\right]\right|^{1-\beta} \\
&\le C\left|\mathbb{E}\left[\sup_{0\le s\le T}\left|\mathbb{E}\left[\sup_{0\le r\le T}\frac{|b_r|}{(T-r)^\beta}\middle|\mathcal{F}_s^0\right]\right|^{\frac{1}{1-\beta}}\middle|\mathcal{F}_t\right]\right|^{1-\beta}.
\end{aligned} \tag{5.18}$$

Thus, Doob's maximal inequality yields that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T} I_1^2\right]\\
&\le C\mathbb{E}\left\{\sup_{0\le t\le T}\left|\mathbb{E}\left[\sup_{0\le s\le T}\left|\mathbb{E}\left[\sup_{0\le r\le T}\frac{|b_r|}{(T-r)^\beta}\middle|\mathcal{F}_s^0\right]\right|^{\frac{1}{1-\beta}}\middle|\mathcal{F}_t\right]\right|^{2(1-\beta)}\right\}\\
&\le C\mathbb{E}\left[\sup_{0\le s\le T}\mathbb{E}\left[\sup_{0\le r\le T}\frac{|b_r|}{(T-r)^\beta}\middle|\mathcal{F}_s^0\right]\right]^2\\
&\le C\mathbb{E}\left[\sup_{0\le r\le T}\left|\frac{|b_r|}{(T-r)^\beta}\right|^2\right].
\end{aligned}\tag{5.19}$$

For the second term which is denoted by I_2, by applying Doob's maximal inequality twice, we have

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le t\le T} I_2^2\right] &\le C\mathbb{E}\left[\sup_{0\le t\le T}\left|\mathbb{E}\left[\int_0^T\mathbb{E}\left[\frac{|b_s|}{(T-s)^\beta}\middle|\mathcal{F}_s^0\right]ds\middle|\mathcal{F}_t\right]\right|^2\right]\\
&\le C\mathbb{E}\left|\int_0^T E\left(\frac{|b_s|}{(T-s)^\beta}\middle|\mathcal{F}_s^0\right)ds\right|^2\\
&\le C\mathbb{E}\left|\sup_{0\le s\le T}\mathbb{E}\left(\sup_{0\le t\le T}\frac{|b_t|}{(T-t)^\beta}|\mathcal{F}_s^0\right)\right|^2\\
&\le C\mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{|b_t|}{(T-t)^\beta}\right|^2\right].
\end{aligned}\tag{5.20}$$

For the third term which is denoted by I_3, Doob's maximal inequality and Hölder inequality imply that

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le t\le T} I_2^2\right] &\le C\mathbb{E}\left[\sup_{0\le t\le T}\left[\mathbb{E}\left(\int_0^T|\xi_s^0|^{\frac{1}{1-\beta}}ds\middle|\mathcal{F}_t\right)\right]^{2(1-\beta)}\right]\\
&\le C\mathbb{E}\left|\int_0^T|\xi_s^0|^{\frac{1}{1-\beta}}ds\right|^{2(1-\beta)}\\
&\le C\mathbb{E}\left[\int_0^T|\xi_s^0|^2\,ds\right].
\end{aligned}\tag{5.21}$$

By (5.19), (5.20) and (5.21), we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{|B_t|}{(T-t)^\beta}\right|^2\right]\\
&\le C+C\mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{|b_t|}{(T-t)^\beta}\right|^2\right]+C\mathbb{E}\left[\int_0^T|\xi_s^0|^2\,ds\right].
\end{aligned}\tag{5.22}$$

By Lemma 5.1.1, we have $Z \in L^2_{\mathbb{F}}(\Omega \times [0,T]; \mathbb{R}^m)$. Thus, it defines a mapping from $\overline{\mathcal{H}}_\alpha \times \overline{\mathcal{H}}_\beta \times L^2_{\mathbb{F}}(\Omega \times [0,T]; \mathbb{R}^m)$ to itself:

$$\Gamma : (x, b, z) \to (X, B, Z).$$

In the following it is sufficient to show Γ is a contraction. From (5.14) we have

$$\frac{|X_t - X'_t|}{(T-t)^\alpha} \leq C \int_0^t \frac{|b_s - b'_s|}{(T-s)^\alpha}\, ds$$

and

$$\begin{aligned}
\frac{|B_t - B'_t|}{(T-t)^\beta} &\leq \frac{C}{(T-t)^\beta} \mathbb{E}\left[\int_t^T \mathbb{E}\left[\frac{|X_s - X'_s|}{T-s} \middle| \mathcal{F}^0_s \right] ds \middle| \mathcal{F}_t \right] \\
&\quad + C\mathbb{E}\left[\int_t^T \mathbb{E}\left[\frac{|b_s - b'_s|}{(T-s)^\beta} \middle| \mathcal{F}^0_s \right] ds \middle| \mathcal{F}_t \right] \\
&\leq \frac{C}{(T-t)^\beta} \mathbb{E}\left[\int_t^T (T-s)^{\alpha-1} \mathbb{E}\left[\int_0^s \frac{|b_r - b'_r|}{(T-r)^\alpha}\, dr \middle| \mathcal{F}^0_s \right] ds \middle| \mathcal{F}_t \right] \\
&\quad + C\mathbb{E}\left[\int_t^T \mathbb{E}\left[\frac{|b_s - b'_s|}{(T-s)^\beta} \middle| \mathcal{F}^0_s \right] ds \middle| \mathcal{F}_t \right].
\end{aligned}$$

By the same argument as that in (5.19), (5.20) and (5.21), there exists some positive constant γ_0 such that

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} \left| \frac{|B_t - B'_t|}{(T-t)^\beta} \right|^2 \right] \leq C T^{\gamma_0} \mathbb{E}\left[\sup_{0 \leq t \leq T} \left| \frac{|b_t|}{(T-t)^\beta} \right|^2 \right].$$

By Lemma 5.1.1, we know that

$$\mathbb{E}\left[\int_0^T |Z^B_t - Z^{B'}_t|^2\, dt \right] \leq \mathbb{E}\left[\sup_{0 \leq t \leq T} \left| \frac{|X_t - X'_t|}{(T-t)^\alpha} \right|^2 \right] + \mathbb{E}\left[\sup_{0 \leq t \leq T} \left| \frac{|B_t - B'_t|}{(T-t)^\beta} \right|^2 \right].$$

Thus, when T is small, Γ is a contraction. □

Corollary 5.1.3. *The mapping $\xi^0 \to AX + B$ is well defined and convex.*

Proof. By Theorem 5.1.2, for each $\xi^0 \in L^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R})$, there exist unique processes X and B. Thus, the mapping $\xi^0 \to AX + B$ is well defined. Moreover, X and B are convex in ξ^0,[2] which is due to the uniqueness in Theorem 5.1.2. □

We are now going to verify the optimaltiy of the candidate. The proof is similar to that of Theorem 4.1.7 in Chapter 4.

[2]Because the initial condition of X leads to a nonhomogeneous term, the mappings $\xi^0 \mapsto X(\xi^0)$ and $\xi^0 \mapsto B(\xi^0)$ cannot be linear generally.

Theorem 5.1.4. *Under the Assumption 5.0.1 and $T < T_1$, for any fixed $\xi^0 \in L^2_{\mathbb{F}^0}(\Omega \times [0,T];\mathbb{R})$, μ^* is the unique solution to MFG of the follower's problem, where*

$$\mu_t^* = \mathbb{E}\left[\frac{A_t X_t + B_t}{2\eta_t}\middle|\mathcal{F}_t^0\right], \quad a.s.\ a.e., \tag{5.23}$$

with X and B the unique solution to (5.9).

Proof. The proof is separated into two steps.

Step 1. The candidate (5.7) is admissible. By Theorem 5.1.2, we have $X_T = 0$. Moreover, Assumption 5.0.1(3) implies $AX + B \in L^2_{\mathbb{F}}(\Omega \times [0,T];\mathbb{R})$. Thus, $(AX + B)/2\eta \in \mathcal{A}_{\mathbb{F}}(0,x)$.

Step 2. The candidate strategy (5.7) is optimal.

For any $0 \le t \le \widetilde{T} < T$, let

$$J(\xi^*;\xi^0,\mu^*,t,x,\widetilde{T}) = \mathbb{E}\left[\int_t^{\widetilde{T}} \kappa_s\mu_s^* X_s^* + \kappa_s^0 X_s^* \xi_s^0 + \eta_s(\xi_s^*)^2 + \lambda_s(X_s^*)^2\,ds\middle|\mathcal{F}_t\right],$$

together with (5.6), one has

$$J(\xi^*;\xi^0,\mu^*,t,x,\widetilde{T}) = \mathbb{E}\left[\int_t^{\widetilde{T}} H(\xi_s^*, X_s^*, Y_s;\xi^0,\mu^*) + \xi_s^* Y_s\,ds\middle|\mathcal{F}_t\right].$$

Thus, for any $\xi \in \mathcal{A}_{\mathbb{F}}(t,x)$, we have

$$\begin{aligned}
&J(\xi;\xi^0,\mu^*,t,x,\widetilde{T}) - J(\xi^*;\xi^0,\mu^*,t,x,\widetilde{T})\\
&= \mathbb{E}\left[\int_t^{\widetilde{T}} H(\xi_s, X_s, Y_s;\xi^0,\mu^*) - H(\xi_s^*, X_s^*, Y_s;\xi^0,\mu^*) + (\xi_s - \xi_s^*)Y_s\,ds\middle|\mathcal{F}_t\right]\\
&= \mathbb{E}\left[\int_t^{\widetilde{T}} H(\xi_s, X_s, Y_s;\xi^0,\mu^*) - H(\xi^*, X_s, Y_s;\xi^0,\mu^*)\right.\\
&\quad \left. + H(\xi^*, X_s, Y_s;\xi^0,\mu^*) - H(\xi_s^*, X_s^*, Y_s;\xi^0,\mu^*) + (\xi_s - \xi_s^*)Y_s\,ds\middle|\mathcal{F}_t\right]\\
&\ge \mathbb{E}\left[\int_t^{\widetilde{T}} H_X(\xi_s^*, X_s^*, Y_s;\xi^0,\mu^*)(X_s - X_s^*) + (\xi_s - \xi_s^*)Y_s\,ds\middle|\mathcal{F}_t\right]\\
&\ge \mathbb{E}\left[\int_t^{\widetilde{T}} (\kappa_s\mu_s^* + \kappa_s^0\xi_s^0 + 2\lambda_s X_s^*)(X_s - X_s^*) + (\xi_s - \xi_s^*)Y_s\middle|\mathcal{F}_t\right].
\end{aligned}$$

Doing integration by part for $(X - X^*)Y$ on $[t,\widetilde{T}]$, one has

$$\mathbb{E}\left[(X_{\widetilde{T}} - X_{\widetilde{T}}^*)Y_{\widetilde{T}}|\mathcal{F}_t\right]$$

$$= -\mathbb{E}\left[\int_t^{\widetilde{T}} (X_s - X_s^*)(\kappa_s\mu_s^* + \kappa_s^0\xi_s^0 + 2\lambda_s X_s^*) + (\xi_s - \xi_s^*)Y_s\, ds \,\middle|\, \mathcal{F}_t\right].$$

Thus, we have

$$J(\xi;\xi^0,\mu^*,t,x,\widetilde{T}) - J(\xi^*;\xi^0,\mu^*,t,x,\widetilde{T}) \geq -\mathbb{E}\left[(X_{\widetilde{T}} - X_{\widetilde{T}}^*)Y_{\widetilde{T}}|\mathcal{F}_t\right].$$

Assumption 5.0.1(3) and Theorem 5.1.2 yield that

$$\lim_{\widetilde{T}\nearrow T} \mathbb{E}\left[(X_{\widetilde{T}} - X_{\widetilde{T}}^*)Y_{\widetilde{T}}|\mathcal{F}_t\right] = 0,$$

which implies that

$$J(\xi;\xi^0,\mu^*,t,x,T) - J(\xi^*;\xi^0,\mu^*,t,x,T) \geq 0.$$

□

5.2. Leader's Problem

The leader chooses a strategy to minimize her cost, taking the feedback through the follower's strategy into account. Hence, there are three states for the leader's problem, that is, (X^0, X, B) which are all influenced by the leader's strategy ξ^0. In view of the cost functional (5.5), the leader's problem is given by a McKean-Vlasov type control problem as follows

$$\begin{cases} \inf_{\xi^0} J^0(\xi^0) \text{ subject to} \\ dX_t = -\dfrac{A_tX_t + B_t}{2\eta_t}\, dt, \\ -dB_t = \left(-\dfrac{A_tB_t}{2\eta_t} + \kappa_t\mathbb{E}\left[\dfrac{A_tX_t + B_t}{2\eta_t}\,\middle|\, \mathcal{F}_t^0\right] + \kappa_t^0\xi_t^0\right) dt - Z_t\, d\overline{W}_t, \\ dX_t^0 = -\xi_t^0\, dt, \\ X_0 = x,\ X_0^0 = x^0,\ B_T = 0, \\ \text{and the liquidation constraint } X_T = X_T^0 = 0, \end{cases} \tag{5.24}$$

where $J^0(\xi^0)$ is defined in (5.5) with μ^* being the equilibrium of the follower's problem as in (5.23).

Let us define the Hamiltonian for leader's problem:

$$\begin{aligned} & H(t, p_t, q_t, r_t, X_t^0, B_t, X_t, \xi_t^0) \\ = & -\xi_t^0 p_t - \left(-\frac{A_tB_t}{2\eta_t} + \kappa_t\mathbb{E}\left[\frac{A_tX_t + B_t}{2\eta_t}\,\middle|\, \mathcal{F}_t^0\right] + \kappa_t^0\xi_t^0\right) q_t - \left(\frac{A_t}{2\eta_t}X_t + \frac{B_t}{2\eta_t}\right) r_t \\ & + \kappa_t\mathbb{E}\left[\frac{A_tX_t + B_t}{2\eta_t}\,\middle|\, \mathcal{F}_t^0\right] X_t^0 + \kappa_t^0 X_t^0\xi_t^0 + \eta_t^0(\xi_t^0)^2 + \lambda_t^0(X_t^0)^2 \\ & + \overline{\lambda}_t\mathbb{E}\left[\frac{A_tX_t + B_t}{2\eta_t}\,\middle|\, \mathcal{F}_t^0\right]^2. \end{aligned} \tag{5.25}$$

It yields the following candidate of leader's optimal strategy

$$\xi^{0,*} = \frac{p + \kappa^0 q - \kappa^0 X^0}{2\eta^0}. \tag{5.26}$$

The maximum principle of mean field type control reduces the problem to the solvability of the following singular FBSDE

$$\begin{cases} dX_t^0 = -\,\xi_t^{0,*}\,dt, \\ dX_t = -\left(\dfrac{A_t}{2\eta_t}X_t + \dfrac{1}{2\eta_t}B_t\right)dt, \\ -dq_t = \left(\dfrac{A_t}{2\eta_t}q_t - \dfrac{1}{2\eta_t}r_t - \mathbb{E}[\kappa_t q_t - \kappa_t X_t^0|\mathcal{F}_t^0]\dfrac{1}{2\eta_t} + \mathbb{E}[\overline{\lambda}_t|\mathcal{F}_t^0]\dfrac{\mu_t^*}{\eta_t}\right)dt, \\ -dp_t = \left(\kappa_t^0\xi_t^{0,*} + 2\lambda_t^0 X_t^0 + \kappa_t\mu_t^*\right)dt - Z_t^p\,d\overline{W}_t, \\ -dr_t = \left(-\dfrac{A_t}{2\eta_t}r_t - \mathbb{E}[\kappa_t q_t - \kappa_t X_t^0|\mathcal{F}_t^0]\dfrac{A_t}{2\eta_t} + \mathbb{E}\left[\overline{\lambda}_t|\mathcal{F}_t^0\right]\dfrac{A_t\mu_t^*}{\eta_t}\right)dt - Z_t^r\,d\overline{W}_t, \\ -dB_t = \left(-\dfrac{A_tB_t}{2\eta_t} + \kappa_t\mu_t^* + \kappa_t^0\xi_t^{0,*}\right)dt - Z_t^B\,d\overline{W}_t, \\ X_0^0 = x^0,\ X_T^0 = 0,\ X_0 = x,\ X_T = 0,\ q_0 = 0,\ B_T = 0. \end{cases} \tag{5.27}$$

To solve the system (5.27), we make the following ansatz

$$p_t = \overline{A}_t X_t^{0,*} + \overline{p}_t \quad \text{and} \quad r_t = \frac{\overline{r}_t}{T-t},$$

from which we have

$$\begin{cases} -d\overline{A}_t = \left(-\dfrac{(\overline{A}_t)^2}{2\eta_t^0} + \dfrac{\kappa_t^0\overline{A}_t}{2\eta_t^0} + 2\lambda_t^0\right)dt - Z_t^{\overline{A}}\,d\overline{W}_t, \\ \overline{A}_T = \infty, \end{cases} \tag{5.28}$$

$$\begin{cases} -d\overline{p}_t = \left(-\dfrac{\overline{A}_t\overline{p}_t}{2\eta_t^0} - \dfrac{\kappa_t^0\overline{A}_t q_t}{2\eta_t^0} + \kappa_t^0\xi_t^{0,*} + \kappa_t\mu_t^*\right)dt - Z_t^{\overline{B}}\,d\overline{W}_t, \\ \overline{p}_T = 0 \end{cases} \tag{5.29}$$

and

$$\begin{cases} -d\overline{r} = \Bigg\{\dfrac{\overline{r}_t}{T-t} - \dfrac{A_t\overline{r}_t}{2\eta_t} - \dfrac{(T-t)A_t}{2\eta_t}\mathbb{E}\left[\kappa_t q_t - \kappa_t X_t^0|\mathcal{F}_t^0\right] \\ \qquad + \dfrac{(T-t)A_t}{\eta_t}\mathbb{E}[\overline{\lambda}_t|\mathcal{F}_t^0]\mu_t^*\Bigg\}dt - Z_t^{\overline{r}}\,d\overline{W}_t, \\ \overline{r}_T = 0. \end{cases} \tag{5.30}$$

Thus, the optimal strategy candidate (5.26) becomes

$$\xi^{0,*} = \frac{\overline{p} + \kappa^0 q + (\overline{A} - \kappa^0)X^0}{2\eta^0} \tag{5.31}$$

and the system (5.27) reduces to the following singular FBSDE

$$
\begin{cases}
dX_t^0 = \left(\dfrac{\kappa_t^0 - \overline{A}_t}{2\eta_t^0} X_t^0 - \dfrac{1}{2\eta_t^0}\overline{p}_t - \dfrac{\kappa_t^0}{2\eta_t^0} q_t\right) dt, \\
dX_t = -\left(\dfrac{A_t}{2\eta_t} X_t + \dfrac{1}{2\eta_t} B_t\right) dt, \\
-dq_t = \left(\dfrac{A_t}{2\eta_t} q_t - \dfrac{1}{2\eta_t}\dfrac{\overline{r}_t}{T-t} - \mathbb{E}[\kappa_t q_t - \kappa_t X_t^0 | \mathcal{F}_t^0]\dfrac{1}{2\eta_t} + \mathbb{E}[\overline{\lambda}_t | \mathcal{F}_t^0]\dfrac{\mu_t^*}{\eta_t}\right) dt, \\
-d\overline{p}_t = \left(\dfrac{\kappa_t^0 - \overline{A}_t}{2\eta_t^0}\overline{p}_t + \kappa_t^0 \dfrac{\kappa_t^0 - \overline{A}_t}{2\eta_t^0} q_t + \kappa_t^0 \dfrac{\overline{A}_t - \kappa_t^0}{2\eta_t^0} X_t^0 + \kappa_t \mu_t^*\right) dt - Z_t^{\overline{p}}\, d\overline{W}_t, \\
-d\overline{r} = \left\{\dfrac{\overline{r}_t}{T-t} - \dfrac{A_t \overline{r}_t}{2\eta_t} - \dfrac{(T-t)A_t}{2\eta_t}\mathbb{E}\left[\kappa_t q_t - \kappa_t X_t^0 | \mathcal{F}_t^0\right] \right. \\
\qquad \left. + \dfrac{(T-t)A_t}{\eta_t}\mathbb{E}[\overline{\lambda}_t | \mathcal{F}_t^0]\mu_t^*\right\} dt - Z_t^{\overline{r}}\, d\overline{W}_t, \\
-dB_t = \left(\dfrac{\kappa_t^0 \overline{A}_t - (\kappa_t^0)^2}{2\eta_t^0} X_t^0 + \dfrac{\kappa_t^0}{2\eta_t^0}\overline{p}_t + \dfrac{(\kappa_t^0)^2}{2\eta_t^0} q_t - \dfrac{A_t}{2\eta_t} B_t + \kappa_t \mu_t^*\right) dt - Z_t^B\, d\overline{W}_t, \\
X_0^0 = X^0,\ X_0 = X,\ q_0 = 0,\ \overline{p}_T = 0,\ \overline{r}_T = 0,\ B_T = 0.
\end{cases}
\tag{5.32}
$$

The same as Chapter 4, for any $\epsilon \in \mathbb{R}$, we introduce the following space

$$
\mathcal{H}_\epsilon := \{Y : \operatorname*{ess\,sup}_{\Omega\times[0,T]} (T-t)^{-\epsilon}|Y_t| < \infty\}
$$

with the norm

$$
\|Y\|_{\mathcal{H}_\epsilon} := \|Y\|_\epsilon := \operatorname*{ess\,sup}_{\Omega\times[0,T]} (T-t)^{-\epsilon}|Y_t|.
$$

In analogy to Lemma 5.1.1, we have the following estimate for the martingale part. The proof is similar to that of Lemma 4.1.4, so we omit it.

Lemma 5.2.1. *If* (5.32) *admits a solution with* $(X^0, X, q, \overline{p}, \overline{r}, B) \in \mathcal{H}_\alpha \times \mathcal{H}_\alpha \times \mathcal{H}_\beta \times \mathcal{H}_\beta \times \mathcal{H}_\alpha \times \mathcal{H}_\beta$*, we have the following estimates*

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T |Z_t^{\overline{p}}|^2\, dt\right] + \mathbb{E}\left[\int_0^T |Z_t^{\overline{r}}|^2\, dt\right] + \mathbb{E}\left[\int_0^T |Z_t^B|^2\, dt\right] \\
&\le C\left(\|X^0\|_\alpha + \|X\|_\alpha + \|B\|_\beta + \|\overline{p}\|_\beta + \|q\|_\beta + \|\overline{r}\|_\alpha\right)
\end{aligned}
$$

Theorem 5.2.2. *There exists* $T_2 \le T_1$ *such that when* $T < T_2$*, there exists a unique solution to* (5.32) *with* $(X^0, X, q, \overline{p}, \overline{r}, B) \in \mathcal{H}_\alpha \times \mathcal{H}_\alpha \times \mathcal{H}_\beta \times \mathcal{H}_\beta \times \mathcal{H}_\alpha \times \mathcal{H}_\beta$.

Proof. For any $(\widetilde{X}^0, \widetilde{X}, \widetilde{q}, \widetilde{\overline{B}}, \widetilde{\overline{r}}, \widetilde{B}) \in \mathcal{H}_\alpha \times \mathcal{H}_\alpha \times \mathcal{H}_\beta \times \mathcal{H}_\beta \times \mathcal{H}_\alpha \times \mathcal{H}_\beta$, we consider the following system

$$
\begin{cases}
dX_t^0 = \left(\dfrac{\kappa_t^0 - \overline{A}_t}{2\eta_t^0} X_t^0 - \dfrac{1}{2\eta_t^0} \widetilde{\overline{p}}_t - \dfrac{\kappa_t^0}{2\eta_t^0} \widetilde{q}_t \right) dt, \\
dX_t = - \left(\dfrac{A_t}{2\eta_t} X_t + \dfrac{1}{2\eta_t} \widetilde{B}_t \right) dt, \\
-dq_t = \Bigg(\dfrac{A_t}{2\eta_t} q_t - \dfrac{1}{2\eta_t} \dfrac{\widetilde{\overline{r}}_t}{T-t} - \mathbb{E}[\kappa_t \widetilde{q}_t - \kappa_t \widetilde{X}_t^0 | \mathcal{F}_t^0] \dfrac{1}{2\eta_t} \\
\qquad + \mathbb{E}[\overline{\lambda}_t | \mathcal{F}_t^0] \mathbb{E}\left[\dfrac{A_t \widetilde{X}_t + \widetilde{B}_t}{2\eta_t} \middle| \mathcal{F}_t^0 \right] \dfrac{1}{\eta_t} \Bigg) dt, \\
-d\overline{p}_t = \Bigg(\dfrac{\kappa_t^0 - \overline{A}_t}{2\eta_t^0} \overline{p}_t + \kappa_t^0 \dfrac{\kappa_t^0 - \overline{A}_t}{2\eta_t^0} q_t + \kappa_t^0 \dfrac{\overline{A}_t - \kappa_t^0}{2\eta_t^0} \widetilde{X}_t^0 \\
\qquad + \kappa_t \mathbb{E}\left[\dfrac{A_t \widetilde{X}_t + \widetilde{B}_t}{2\eta_t} \middle| \mathcal{F}_t^0 \right] \Bigg) dt - Z_t^{\overline{p}}\, d\overline{W}_t, \\
-d\overline{r} = \Bigg(\dfrac{\overline{r}_t}{T-t} - \dfrac{A_t \overline{r}_t}{2\eta_t} - \dfrac{(T-t)A_t}{2\eta_t} \mathbb{E}\left[\kappa_t \widetilde{q}_t - \kappa_t \widetilde{X}_t^0 | \mathcal{F}_t^0 \right] \\
\qquad + \dfrac{(T-t)A_t}{\eta_t} \mathbb{E}[\overline{\lambda}_t | \mathcal{F}_t^0] \mathbb{E}\left[\dfrac{A_t X_t + \widetilde{B}_t}{2\eta_t} \middle| \mathcal{F}_t^0 \right] \Bigg) dt - Z_t^{\overline{r}}\, d\overline{W}_t, \\
-dB_t = \Bigg(\dfrac{\kappa_t^0 \overline{A}_t - (\kappa_t^0)^2}{2\eta_t^0} \widetilde{X}_t^0 + \dfrac{\kappa_t^0}{2\eta_t^0} \widetilde{\overline{p}}_t + \dfrac{(\kappa_t^0)^2}{2\eta_t^0} \widetilde{q}_t - \dfrac{A_t}{2\eta_t} \widetilde{B}_t \\
\qquad + \kappa_t \mathbb{E}\left[\dfrac{A_t \widetilde{X}_t + \widetilde{B}_t}{2\eta_t} | \mathcal{F}_t^0 \right] \Bigg) dt - Z_t^B\, d\overline{W}_t, \\
X_0^0 = x^0,\ X_0 = x,\ q_0 = 0,\ \overline{p}_T = 0,\ \overline{r}_T = 0,\ B_T = 0.
\end{cases}
$$

Thus, it can be checked directly that

$$
(X^0, X, q, \overline{B}, \overline{r}, B) \in \mathcal{H}_\alpha \times \mathcal{H}_\alpha \times \mathcal{H}_\beta \times \mathcal{H}_\beta \times \mathcal{H}_\alpha \times \mathcal{H}_\beta.
$$

Thus, we get a mapping from $\mathcal{H}_\beta \times \mathcal{H}_\beta \times \mathcal{H}_\alpha \times \mathcal{H}_\alpha \times \mathcal{H}_\beta \times \mathcal{H}_\alpha$ to itself. It is sufficient to show this mapping is a contraction.

For X^0-component, we have

$$
\begin{aligned}
|X_t^0 - X_t^{0'}| &\le C(T-t)^\alpha \int_0^t \left(\frac{|\widetilde{\overline{p}}_s - \widetilde{\overline{p}}'_s|}{(T-s)^\alpha} + \frac{|\widetilde{q}_s - \widetilde{q}'_s|}{(T-s)^\alpha} \right) ds \\
&\le C(T-t)^\alpha T^{\beta+1-\alpha} \|\widetilde{\overline{p}} - \widetilde{\overline{p}}'\|_\beta + (T-t)^\alpha T^{\beta+1-\alpha} \|\widetilde{q} - \widetilde{q}'\|_\beta.
\end{aligned}
\tag{5.33}
$$

For X-component, we have

$$
|X_t - X'_t| \le (T-t)^\alpha T^{\beta+1-\alpha} \|\widetilde{B} - \widetilde{B}'\|_\beta. \tag{5.34}
$$

For q-component, we have

$$
|q_t - q'_t| \le C(T-t)^\beta \int_0^t \frac{|\widetilde{\overline{r}}_s - \widetilde{\overline{r}}'_s|}{(T-s)^{\beta+1}} ds + C(T-t)^\beta \int_0^t \operatorname*{ess\,sup}_\Omega \frac{|\widetilde{q}_s - \widetilde{q}'_s|}{(T-s)^\beta} ds
$$

$$
\begin{aligned}
&+ C(T-t)^{\alpha}\int_0^t \operatorname*{ess\,sup}_{\Omega} \frac{|\widetilde{X}^0_s - \widetilde{X}^{0'}_s|}{(T-s)^{\alpha}}\,ds + C(T-t)^{\beta}\int_0^t \operatorname*{ess\,sup}_{\Omega} \frac{|\widetilde{B}_s - \widetilde{B}'_s|}{(T-s)^{\beta}}\,ds \\
&+ C(T-t)^{\beta}\int_0^t \operatorname*{ess\,sup}_{\Omega} \frac{|X_s - X'_s|}{(T-s)^{1+\beta}}\,ds \\
\le{}& CT^{\alpha-\beta}(T-t)^{\beta}\|\widetilde{\overline{r}} - \widetilde{\overline{r}}'\|_{\alpha} + CT(T-t)^{\beta}\|\widetilde{q} - \widetilde{q}'\|_{\beta} \\
&+ CT(T-t)^{\alpha}\|\widetilde{X}^0 - \widetilde{X}^{0'}\|_{\alpha} + CT^{\alpha-\beta}(T-t)^{\beta}\|\widetilde{X} - \widetilde{X}'\|_{\alpha} \\
&+ CT(T-t)^{\beta}\|\widetilde{B} - \widetilde{B}'\|_{\beta}.
\end{aligned} \tag{5.35}
$$

For $\overline{p}$-component, we have

$$
\begin{aligned}
|\overline{p}_t - \overline{p}'_t| \le{}& C\int_t^T \operatorname*{ess\,sup}_{\Omega} |q_s - q'_s|\,ds + C\int_t^T \operatorname*{ess\,sup}_{\Omega} \frac{|q_s - q'_s|}{T-s}\,ds \\
&+ C\int_t^T \operatorname*{ess\,sup}_{\Omega} |\widetilde{X}^0_s - \widetilde{X}^{0'}_s|\,ds + C\int_t^T \operatorname*{ess\,sup}_{\Omega} \frac{|\widetilde{X}^0_s - \widetilde{X}^{0'}_s|}{T-s}\,ds \\
&+ C\int_t^T \operatorname*{ess\,sup}_{\Omega} \frac{|\widetilde{X}_s - \widetilde{X}'_s|}{T-s}\,ds + C\int_t^T \operatorname*{ess\,sup}_{\Omega} |\widetilde{B}_s - \widetilde{B}'_s|\,ds,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\frac{|\overline{p}_t - \overline{p}'_t|}{(T-t)^{\beta}} \\
\le{}& C\int_t^T \operatorname*{ess\,sup}_{\Omega} \frac{|q_s - q'_s|}{(T-s)^{\beta}}\,ds + \frac{C}{(T-t)^{\beta}}\int_t^T \operatorname*{ess\,sup}_{\Omega} \frac{|q_s - q'_s|}{T-s}\,ds \\
&+ \frac{C}{(T-t)^{\beta}}\int_t^T \operatorname*{ess\,sup}_{\Omega} |\widetilde{X}^0_s - \widetilde{X}^{0'}_s|\,ds + \frac{C}{(T-t)^{\beta}}\int_t^T \operatorname*{ess\,sup}_{\Omega} \frac{|\widetilde{X}^0_s - \widetilde{X}^{0'}_s|}{T-s}\,ds \\
&+ \frac{C}{(T-t)^{\beta}}\int_t^T \operatorname*{ess\,sup}_{\Omega} \frac{|\widetilde{X}_s - \widetilde{X}'_s|}{T-s}\,ds + C\int_t^T \operatorname*{ess\,sup}_{\Omega} \frac{|\widetilde{B}_s - \widetilde{B}'_s|}{(T-s)^{\beta}}\,ds \\
\le{}& CT\|q - q'\|_{\beta} + C\|q - q'\|_{\beta} + CT^{\alpha+1-\beta}\|\widetilde{X}^0 - \widetilde{X}^{0'}\|_{\alpha} + CT^{\alpha-\beta}\|\widetilde{X}^0 - \widetilde{X}^{0'}\|_{\alpha} \\
&+ CT^{\alpha-\beta}\|\widetilde{X} - \widetilde{X}'\|_{\alpha} + CT\|\widetilde{B} - \widetilde{B}'\|_{\beta}.
\end{aligned} \tag{5.36}
$$

For $\overline{r}$-component, we have

$$
\begin{aligned}
|\overline{r}_t - \overline{r}'_t| \le{}& \int_t^T \frac{(T-s)^{2\alpha-1}}{(T-t)^{\alpha}} \operatorname*{ess\,sup}_{\Omega} \frac{|\overline{r}_s - \overline{r}'_s|}{(T-s)^{\alpha}}\,ds \\
&+ C\int_t^T \left(\operatorname*{ess\,sup}_{\Omega} |\widetilde{q}_s - \widetilde{q}'_s| + \operatorname*{ess\,sup}_{\Omega} |\widetilde{X}^0_s - \widetilde{X}^{0'}_s|\right) ds \\
&+ C\int_t^T \operatorname*{ess\,sup}_{\Omega} \frac{|X_s - X'_s|}{T-s}\,ds + C\int_t^T \operatorname*{ess\,sup}_{\Omega} |\widetilde{B}_s - \widetilde{B}'_s|\,ds \qquad (5.37) \\
\le{}& \frac{1}{2\alpha}\|\overline{r} - \overline{r}'\|_{\alpha,T}(T-t)^{\alpha} + C(T-t)^{\beta+1}\|\widetilde{q} - \widetilde{q}'\|_{\beta} \\
&+ C(T-t)^{\alpha+1}\|\widetilde{X}^0 - \widetilde{X}^{0'}\|_{\alpha} + C(T-t)^{\alpha}\|X - X'\|_{\alpha} \\
&+ C(T-t)^{\beta+1}\|\widetilde{B} - \widetilde{B}'\|_{\beta}.
\end{aligned}
$$

For B-component, we have

$$
\begin{aligned}
|B_t - B_t'| &\le C\int_t^T \operatorname*{ess\,sup}_{\Omega} |\widetilde{X}^0_s - \widetilde{X}^{0'}_s|\, ds + C\int_t^T \operatorname*{ess\,sup}_{\Omega} \frac{|\widetilde{X}^0_s - \widetilde{X}^{0'}_s|}{T-s}\, ds \\
&\quad + C\int_t^T \operatorname*{ess\,sup}_{\Omega} |\widetilde{\overline{p}}_s - \widetilde{\overline{p}}'_s|\, ds + C\int_t^T \operatorname*{ess\,sup}_{\Omega} |\widetilde{q}_s - \widetilde{q}'_s|\, ds \\
&\quad + C\int_t^T \operatorname*{ess\,sup}_{\Omega} \frac{|\widetilde{X}_s - \widetilde{X}'_s|}{T-s}\, ds + \frac{\|\kappa\|}{2\underline{\eta}}\int_t^T \operatorname*{ess\,sup}_{\Omega} |\widetilde{B}_s - \widetilde{B}'_s|\, ds \\
&\le CT(T-t)^\beta \|\widetilde{X}^0 - \widetilde{X}^{0'}\|_\alpha + C(T-t)^\alpha \|\widetilde{X}^0 - \widetilde{X}^{0'}\|_\alpha \\
&\quad + CT(T-t)^\beta \|\widetilde{\overline{p}} - \widetilde{\overline{p}}'\|_\beta + CT(T-t)^\beta \|\widetilde{q} - \widetilde{q}'\|_\beta \\
&\quad + C(T-t)^\alpha \|\widetilde{X} - \widetilde{X}'\|_\alpha + CT(T-t)^\beta \|\widetilde{B} - \widetilde{B}'\|_\beta.
\end{aligned} \tag{5.38}
$$

From (5.33), (5.34), (5.35), (5.36), (5.37) and (5.38), there exists a positive constant γ_0 such that

$$
\begin{aligned}
&\|X^0 - X^{0'}\|_\alpha + \|X - X'\|_\alpha + \|B - B'\|_\beta + \|\overline{p} - \overline{p}'\|_\beta + \|q - q'\|_\beta + \|\overline{r} - \overline{r}'\|_\alpha \\
\le\ & CT^{\gamma_0}\Big(\|\widetilde{X}^0 - \widetilde{X}^{0'}\|_\alpha + \|\widetilde{X} - \widetilde{X}'\|_\alpha + \|\widetilde{B} - \widetilde{B}'\|_\beta \\
&+ \|\widetilde{\overline{p}} - \widetilde{\overline{p}}'\|_\beta + \|\widetilde{q} - \widetilde{q}'\|_\beta + \|\widetilde{\overline{r}} - \widetilde{\overline{r}}'\|_\alpha\Big).
\end{aligned}
$$

By Lemma 5.2.1, we have as well

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T |Z^{\overline{p}}|^2\, dt\right] + \mathbb{E}\left[\int_0^T |Z^{\overline{r}}|^2\, dt\right] + \mathbb{E}\left[\int_0^T |Z^{B}|^2\, dt\right] \\
\le\ & CT^{\gamma_0}\Big(\|\widetilde{X}^0 - \widetilde{X}^{0'}\|_\alpha + \|\widetilde{X} - \widetilde{X}'\|_\alpha + \|\widetilde{B} - \widetilde{B}'\|_\beta \\
&+ \|\widetilde{\overline{p}} - \widetilde{\overline{p}}'\|_\beta + \|\widetilde{q} - \widetilde{q}'\|_\beta + \|\widetilde{\overline{r}} - \widetilde{\overline{r}}'\|_\alpha\Big).
\end{aligned}
$$

Thus, we get the desired result when T is small. □

The next theorem verifies that the candidate (5.26) is the unique optimal strategy of the leader. For this purpose, we make one more assumption, under which, the cost functional is convex.

Assumption 5.2.3. Suppose

$$
\lambda^0 - \frac{\kappa^0}{2} - \frac{\kappa}{2} \ge 0, \quad \eta^0 - \frac{\kappa^0}{2} \ge 0, \quad \overline{\lambda} - \frac{\kappa}{2} \ge 0.
$$

Theorem 5.2.4. *Suppose Assumption 5.0.1 and 5.2.3 hold. Then the candidate given by* (5.26) *is the unique optimal strategy to the McKean-Vlasov control problem* (5.24)*.*

Proof. We denote by $(X^{0,*}, X^*, B^*)$ the states corresponding to the candidate strategy and by (X^0, X, B) the states corresponding to s generic strategy. The verification is split into the following several steps.

Step 1. (5.26) is admissible.

Indeed, by Assumption 5.0.1 and Theorem 5.2.2, it can be checked directly that

$$\xi^{0,*} := \frac{p + \kappa^0 q - \kappa^0 X^{0,*}}{2\eta^0} \in L^2([0,T] \times \Omega), \quad \textit{and} \quad X_T^{0,*} = X_T^* = 0,$$

which implies $\xi^{0,*} \in \mathcal{A}_{\mathbb{F}}(0, x^0)$.

Step 2. J^0 is strictly convex in ξ^0. In fact, the cost functional can be rewritten as follows

$$\begin{aligned} J^0(\xi^0) = \mathbb{E}\Bigg[\int_0^T & \frac{\kappa_t}{2}(X_t^0 + \mu_t)^2 + \frac{\kappa_t^0}{2}(X_t^0 + \xi_t^0)^2 + \left(\lambda_t^0 - \frac{\kappa_t^0}{2} - \frac{\kappa_t}{2}\right)(X_t^0)^2 \\ & + \left(\eta_t^0 - \frac{\kappa_t^0}{2}\right)(\xi_t^0)^2 + \left(\overline{\lambda}_t - \frac{\kappa_t}{2}\right)\mu_t^2 \, dt\Bigg], \end{aligned}$$

where

$$\mu_t = \mathbb{E}\left[\frac{A_t X_t + B_t}{2\eta_t} \middle| \mathcal{F}_t^0\right].$$

By Corollary 5.1.3, μ is convex in ξ^0. Thus, J^0 is strictly convex in ξ^0. As a result, there is at most one optimal strategy for (5.24). Moreover, for fixed candidate strategy and adjoint processes, the Hamiltonian defined by (5.25) is convex in states.

Step 3. Integration by part for $(X^0 - X^{0,*})p$, $(X - X^*)r$ and $(B - B^*)q$. For any $0 \le \widetilde{T} < T$, integration by part yields that

$$\begin{aligned} (X_{\widetilde{T}}^0 - X_{\widetilde{T}}^{0,*})p_{\widetilde{T}} = & -\int_0^{\widetilde{T}} (X_t^0 - X_t^{0,*})\left(\kappa_t \mathbb{E}\left[\frac{A_t X_t^* + B_t^*}{2\eta_t} \middle| \mathcal{F}_t^0\right] + \kappa_t^0 \xi_t^{0,*} + 2\lambda_t^0 X_t^{0,*}\right) dt \\ & + \int_0^{\widetilde{T}} (X_t^0 - X_t^{0,*}) Z_t^p \, d\overline{W}_t - \int_0^{\widetilde{T}} p_t(\xi_t^0 - \xi_t^{0,*}) \, dt, \end{aligned} \tag{5.39}$$

$$\begin{aligned} & (X_{\widetilde{T}} - X_{\widetilde{T}}^*)r_{\widetilde{T}} \\ = & -\int_0^{\widetilde{T}} (X_t - X_t^*)\left(-\mathbb{E}\left[\kappa_t q_t - \kappa_t X_t^{0,*} \middle| \mathcal{F}_t^0\right]\frac{A_t}{2\eta_t} + \mathbb{E}[\overline{\lambda}_t | \mathcal{F}_t^0]\mathbb{E}\left[\frac{A_t X_t^* + B_t^*}{2\eta_t} \middle| \mathcal{F}_t^0\right]\frac{A_t}{\eta_t}\right) dt \\ & + \int_0^{\widetilde{T}} (B_t - B_t^*) Z_t^r \, d\overline{W}_t - \int_0^{\widetilde{T}} \frac{1}{2\eta_t} r_t (B_t - B_t^*) \, dt \end{aligned} \tag{5.40}$$

and

$$
\begin{aligned}
&(B_{\widetilde{T}} - B^*_{\widetilde{T}})q_{\widetilde{T}} \\
= & -\int_0^{\widetilde{T}} (B_t - B^*_t)\left(-\frac{r_t}{2\eta_t} - \mathbb{E}[\kappa_t q_t - \kappa_t X^{0,*}_t|\mathcal{F}^0_t]\frac{1}{2\eta_t} + \mathbb{E}[\overline{\lambda}_t|\mathcal{F}^0_t]\mathbb{E}\left[\left.\frac{AX^*_t + B^*_t}{2\eta_t}\right|\mathcal{F}^0_t\right]\frac{1}{\eta_t}\right) dt \\
&+ \int_0^{\widetilde{T}} (B_t - B^*_t)Z^r_t\, d\overline{W}_t \\
&- \int_0^{\widetilde{T}} q_t\left(\kappa_t\mathbb{E}\left[\left.\frac{A_t(X_t - X^*_t) + B_t - B^*_t}{2\eta_t}\right|\mathcal{F}^0_t\right] + \kappa^0_t(\xi^0_t - \xi^{0,*}_t)\right) dt.
\end{aligned}
\tag{5.41}
$$

Based on the observation that

> "for any random variables X_1, X_2 and filtration $\mathcal{G}$,
> one has $\mathbb{E}[X_1\mathbb{E}[X_2|\mathcal{G}]] = \mathbb{E}[\mathbb{E}[X_1|\mathcal{G}]X_2] = \mathbb{E}[\mathbb{E}[X_1|\mathcal{G}]\mathbb{E}[X_2|\mathcal{G}]]$",

taking expectation on both sides of (5.39), (5.40) and (5.41), we have

$$
\begin{aligned}
&\mathbb{E}\left[(X^0_{\widetilde{T}} - X^{0,*}_{\widetilde{T}})p_{\widetilde{T}}\right] + \mathbb{E}\left[(X_{\widetilde{T}} - X^*_{\widetilde{T}})r_{\widetilde{T}}\right] + \mathbb{E}\left[(B_{\widetilde{T}} - B^*_{\widetilde{T}})q_{\widetilde{T}}\right] \\
= & -\mathbb{E}\left[\int_0^{\widetilde{T}} (X^0_t - X^{0,*}_t)\left(\kappa_t\mathbb{E}\left[\left.\frac{A_tX^*_t + B^*_t}{2\eta_t}\right|\mathcal{F}^0_t\right] + \kappa^0_t\xi^{0,*}_t + 2\lambda^0_tX^{0,*}_t\right) dt\right] \\
&- \mathbb{E}\left[\int_0^{\widetilde{T}} p_t(\xi^0_t - \xi^{0,*}_t)\, dt\right] - \mathbb{E}\left[\int_0^{\widetilde{T}} \frac{A_t(X_t - X^*_t)}{2\eta_t}\mathbb{E}[\kappa_tX^{0,*}_t|\mathcal{F}^0_t]\, dt\right] \\
&- \mathbb{E}\left[\int_0^{\widetilde{T}} \mathbb{E}[\overline{\lambda}_t|\mathcal{F}^0_t]\mathbb{E}\left[\left.\frac{A_tX^*_t + B^*_t}{2\eta_t}\right|\mathcal{F}^0_t\right]\frac{A_t(X_t - X^*_t)}{\eta_t}\, dt\right] \\
&- \mathbb{E}\left[\int_0^{\widetilde{T}} \frac{(B_t - B^*_t)}{2\eta_t}\mathbb{E}[\kappa_tX^{0,*}_t|\mathcal{F}^0_t]\, dt\right] - \mathbb{E}\left[\int_0^{\widetilde{T}} \kappa^0_tq_t(\xi^0_t - \xi^{0,*}_t)\, dt\right] \\
&- \mathbb{E}\left[\int_0^{\widetilde{T}} \mathbb{E}[\overline{\lambda}_t|\mathcal{F}^0_t]\mathbb{E}\left[\left.\frac{AX^*_t + B^*_t}{2\eta_t}\right|\mathcal{F}^0_t\right]\frac{B_t - B^*_t}{\eta_t}\, dt\right].
\end{aligned}
\tag{5.42}
$$

Step 4. Optimality of (5.26).
For any $\widetilde{T} < T$, define

$$
\begin{aligned}
&\widetilde{J}^0(\xi^0) \\
:= &\ \mathbb{E}\left[\int_0^{\widetilde{T}} \kappa_t\mathbb{E}\left[\left.\frac{A_tX_t + B_t}{2\eta_t}\right|\mathcal{F}^0_t\right]X^0_t + \kappa^0_t\xi^0_tX^0_t\right. \\
&\left. + \eta^0_t(\xi^0_t)^2 + \lambda^0_t(X^0_t)^2 + \overline{\lambda}_t\left|\mathbb{E}\left[\left.\frac{A_tX_t + B_t}{2\eta_t}\right|\mathcal{F}^0_t\right]\right|^2 dt\right].
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\widetilde{J}^0(\xi^0) - \widetilde{J}^0(\xi^{0,*}) \\
=& \int_0^{\widetilde{T}} \Big[H(t, p_t, r_t, q_t, X_t^0, X_t, B_t, \xi_t^0) - H(t, p_t, r_t, q_t, X_t^{0,*}, X_t^*, B_t^*, \xi_t^{0,*}) \Big] \, dt \\
&+ \int_0^{\widetilde{T}} \Bigg[(\xi_t^0 - \xi_t^{0,*}) p_t + \frac{A_t(X_t - X_t^*) r_t}{2\eta_t} + \frac{(B_t - B_t^*) r_t}{2\eta_t} \\
&+ \left(\kappa_t^0 (\xi_t^0 - \xi_t^{0,*}) - \frac{A_t}{2\eta_t}(B_t - B_t^*) + \kappa_t \mathbb{E}\left[\frac{A_t(X_t - X_t^*) + B_t - B_t^*}{2\eta_t} \middle| \mathcal{F}_t^0 \right] \right) q_t \Bigg] \, dt \\
=& \int_0^{\widetilde{T}} \Big[H(t, p_t, r_t, q_t, X_t^0, X_t, B_t, \xi_t^0) - H(t, p_t, r_t, q_t, X_t^0, X_t, B_t, \xi_t^{0,*}) \Big] \, dt \\
&+ \int_0^{\widetilde{T}} \Big[H(t, p_t, r_t, q_t, X_t^0, X_t, B_t, \xi_t^{0,*}) - H(t, p_t, r_t, q_t, X_t^{0,*}, X_t^*, B_t^*, \xi_t^{0,*}) \Big] \, dt \\
&+ \int_0^{\widetilde{T}} \Bigg[(\xi_t^0 - \xi_t^{0,*}) p_t + \frac{A_t(X_t - X_t^*) r_t}{2\eta_t} + \frac{(B_t - B_t^*) r_t}{2\eta_t} \\
&+ \left(\kappa_t^0 (\xi_t^0 - \xi_t^{0,*}) - \frac{A_t}{2\eta_t}(B_t - B_t^*) + \kappa_t \mathbb{E}\left[\frac{A_t(X_t - X_t^*) + B_t - B_t^*}{2\eta_t} \middle| \mathcal{F}_t^0 \right] \right) q_t \Bigg] \, dt \\
\geq& \int_0^{\widetilde{T}} \Big[H(t, p_t, r_t, q_t, X_t^0, X_t, B_t, \xi_t^{0,*}) - H(t, p_t, r_t, q_t, X_t^{0,*}, X_t^*, B_t^*, \xi_t^{0,*}) \Big] \, dt \\
&+ \int_0^{\widetilde{T}} \Bigg[(\xi_t^0 - \xi_t^{0,*}) p_t + \frac{A_t(X_t - X_t^*) r_t}{2\eta_t} + \frac{(B_t - B_t^*) r_t}{2\eta_t} \\
&+ \left(\kappa_t^0 (\xi_t^0 - \xi_t^{0,*}) - \frac{A_t}{2\eta_t}(B_t - B_t^*) + \kappa_t \mathbb{E}\left[\frac{A_t(X_t - X_t^*) + B_t - B_t^*}{2\eta_t} \middle| \mathcal{F}_t^0 \right] \right) q_t \Bigg] \, dt \\
&\qquad \text{(the candidate (5.26) is a minimizer of } H). \qquad (5.43)
\end{aligned}
$$

The convexity of $H(t, p, r, q, \cdot, \cdot, \cdot, \xi^0)$ (see Step 2) yields that

$$
\begin{aligned}
&H(t, p_t, r_t, q_t, X_t^0, X_t, B_t, \xi_t^{0,*}) - H(t, p_t, r_t, q_t, X_t^{0,*}, X_t^*, B_t^*, \xi_t^{0,*}) \\
\geq& \left(\kappa_t \mathbb{E}\left[\frac{A_t X_t^* + B_t^*}{2\eta_t} \middle| \mathcal{F}_t^0 \right] + \kappa_t^0 \xi_t^{0,*} + 2\lambda_t^0 X_t^{0,*} \right) (X_t^0 - X_t^{0,*}) - \frac{A_t r_t}{2\eta_r}(X_t - X_t^*) \\
&- \kappa_t q_t \mathbb{E}\left[\frac{A_t}{2\eta_t}(X_t - X_t^*) \middle| \mathcal{F}_t^0 \right] + \kappa_t X_t^{0,*} \mathbb{E}\left[\frac{A_t}{2\eta_t}(X_t - X_t^*) \middle| \mathcal{F}_t^0 \right] \\
&+ \overline{\lambda} \mathbb{E}\left[\frac{A_t X_t^* + B_t^*}{2\eta_t} \middle| \mathcal{F}_t^0 \right] \mathbb{E}\left[\frac{A_t}{\eta_t}(X_t - X_t^*) \middle| \mathcal{F}_t^0 \right] + \frac{A_t q_t}{2\eta_t}(B_t - B_t^*) - \frac{r_t}{2\eta_t}(B_t - B_t^*) \\
&+ (\kappa_t X_t^{0,*} - \kappa_t q_t) \mathbb{E}\left[\frac{B_t - B_t^*}{2\eta_t} \middle| \mathcal{F}_t^0 \right] + \overline{\lambda} \mathbb{E}\left[\frac{A_t X_t^* + B_t^*}{2\eta_t} \middle| \mathcal{F}_t^0 \right] \mathbb{E}\left[\frac{B_t - B_t^*}{\eta_t} \middle| \mathcal{F}_t^0 \right]. \\
&\qquad (5.44)
\end{aligned}
$$

Based on the observation that

"for any random variables X_1, X_2 and X_3 and any filtration $\mathcal{G}$, one has $\mathbb{E}[\mathbb{E}[X_1|\mathcal{G}]\mathbb{E}[X_2|\mathcal{G}]X_3] = \mathbb{E}[X_1\mathbb{E}[X_2|\mathcal{G}]\mathbb{E}[X_3|\mathcal{G}]] = \mathbb{E}[\mathbb{E}[X_1|\mathcal{G}]\mathbb{E}[X_2|\mathcal{G}]\mathbb{E}[X_3|\mathcal{G}]]$", taking expectation, plugging (5.44) and (5.42) into (5.43), we have

$$\widetilde{J}^0(\xi^0) - \widetilde{J}^0(\xi^{0,*}) + \mathbb{E}\left[(X^0_{\widetilde{T}} - X^{0,*}_{\widetilde{T}})p_{\widetilde{T}}\right] + \mathbb{E}\left[(X_{\widetilde{T}} - X^*_{\widetilde{T}})r_{\widetilde{T}}\right] + \mathbb{E}\left[(B_{\widetilde{T}} - B^*_{\widetilde{T}})q_{\widetilde{T}}\right] \geq 0.$$

For the last three terms above, Theorem 5.2.2 yields that

$$\begin{aligned}
&\mathbb{E}\left|(X^0_{\widetilde{T}} - X^{0,*}_{\widetilde{T}})p_{\widetilde{T}}\right| \\
&\leq \mathbb{E}\left|\overline{A}_{\widetilde{T}} X^{0,*}_{\widetilde{T}} X^0_{\widetilde{T}} + \overline{A}_{\widetilde{T}}\left(X^{0,*}_{\widetilde{T}}\right)^2 + \overline{p}_{\widetilde{T}}\left|X^0_{\widetilde{T}} - X^{0,*}_{\widetilde{T}}\right|\right| \\
&\leq \frac{C}{2(T-\widetilde{T})}\mathbb{E}\left[\left|\int_{\widetilde{T}}^T \xi^0_t\,dt\right|^2 + 3\left|\int_{\widetilde{T}}^T \xi^{0,*}_t\,dt\right|^2\right] + \mathbb{E}\left|\overline{p}_{\widetilde{T}}\left|X^0_{\widetilde{T}} - X^{0,*}_{\widetilde{T}}\right|\right| \\
&\leq \frac{C}{2}\mathbb{E}\left[\int_{\widetilde{T}}^T (\xi^0_t)^2\,dt + 3\int_{\widetilde{T}}^T (\xi^{0,*}_t)^2\,dt\right] + \mathbb{E}\left|\overline{p}_{\widetilde{T}}\left|X^0_{\widetilde{T}} - X^{0,*}_{\widetilde{T}}\right|\right| \\
&\to 0, \qquad \text{as } \widetilde{T} \uparrow T,
\end{aligned}$$

and similarly,

$$\lim_{\widetilde{T}\uparrow T}\mathbb{E}\left[(X_{\widetilde{T}} - X^*_{\widetilde{T}})r_{\widetilde{T}}\right] = 0, \quad \lim_{\widetilde{T}\uparrow T}\mathbb{E}\left[(B_{\widetilde{T}} - B^*_{\widetilde{T}})q_{\widetilde{T}}\right] = 0.$$

Thus, letting $\widetilde{T} \uparrow T$, dominated convergence yields

$$J^0(\xi^0) - J^0(\xi^{0,*}) \geq 0.$$

□

A. Appendix

A.1. Some useful Lemmas and an Itô's Formula

This subsection states some useful lemmas and Itô formulas, which have been frequently used. The first lemma and corollary are from [Che05].

Lemma A.1.1. *Let $\{a_k, k \in \mathbb{N}\}$ be a sequence of nonnegative numbers satisfying*

$$a_{k+1} \leq C_0 b^k a_k^{1+\delta},$$

where $b > 1$, $\delta > 0$ and C_0 is a positive constant. Then, if $a_0 \leq \theta_0 := C_0^{-\frac{1}{\delta}} b^{-\frac{1}{\delta^2}}$, we have $\lim_{k\to\infty} a_k = 0$.

Corollary A.1.2. *Let $\phi : [r_0, \infty] \to \mathbb{R}^+$ be a nonnegative and decreasing function. Assume there exist constants $C_1 > 0$, $\alpha > 0$ and $\varsigma > 1$ such that for any $r_0 < r < l$,*

$$\phi(l) \leq \frac{C_1}{(l-r)^\alpha} \phi(r)^\varsigma.$$

Then for any d satisfying

$$d \geq C_1^{\frac{1}{\alpha}} |\phi(r_0)|^{\frac{\varsigma-1}{\alpha}} 2^{\frac{\varsigma}{\varsigma-1}},$$

we have $\phi(r_0 + d) = 0$.

The following embedding lemma is from [QT12].

Lemma A.1.3. *If for each $t \in [0, T]$, $u \in \mathcal{V}_2(\mathcal{O}_t)$, then we have*

$$\|u\|_{0,\frac{2(n+2)}{n};\mathcal{O}_t} \leq C \|\nabla u\|_{0,2;\mathcal{O}_t}^{\frac{n}{n+2}} \mathit{esssup}_{(\omega,s)\in\Omega\times[t,T]} \|u(\omega, s)\|^{\frac{n}{n+2}} \leq C \|u\|_{\mathcal{V}_2(\mathcal{O}_t)},$$

where C only depends on n.

Now, we are going to present the Itô formulas, which have been frequently used in the main text. We assume that Φ is a function that satisfies the following properties:

(1) $\Phi \in \mathcal{C}(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R})$ and $\partial_t \Phi(t, x, u)$, $\Phi'(t, x, u)$, $\Phi''(t, x, u)$ and $\partial_j \Phi'(t, x, u)$, $j = 1, 2, \cdots, n$ exist and are continuous;

(2) $\Phi'(t, x, 0) = 0$ for any $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$;

(3) $\sup_{t\in\mathbb{R}^+, x\in\mathbb{R}^n} |\partial_j \Phi'(t, x, u)| \leq C|u|$, $j = 1, 2, \cdots, n$;

(4)
$$\sup_{t\in\mathbb{R}^+,x\in\mathbb{R}^n,u\in\mathbb{R}/\{0\}}\left\{|\Phi''(t,x,u)|+\frac{1}{|u|^2}|\partial_t\Phi(t,x,u)-\partial_t\Phi(t,x,0)|\right\}<\infty,$$
where $\partial_j\Phi(t,x,u)=\partial_{x_j}\Phi(t,x,u)$, $\Phi'(t,x,u)=\partial_u\Phi(t,x,u)$ and $\Phi''(t,x,u)=\partial_u^2\Phi(t,x,u)$.

Suppose that the following BSPDE
$$\begin{cases} -du(t,x)=[\partial_j(a^{ij}\partial_i u(t,x)+\sigma^{jr}v^r(t,x))+\bar{f}(t,x)+\nabla\cdot\bar{g}(t,x)]\,dt \\ \qquad\qquad +\mu(dt,x)-v^r(t,x)\,dW_t^r, \quad (t,x)\in Q, \\ u(T,x)=G(x), \quad x\in\mathcal{O}, \end{cases} \tag{A.1}$$
holds in the weak sense where $(u,v)\in\mathcal{V}_2(Q)\times\mathcal{M}^{0,2}(Q)$, μ is a stochastic regular measure, $\bar{f}$, $\bar{g}$ and G satisfy $(\mathcal{A}_3)$, a and σ satisfy $(\mathcal{A}_2)$.

When Φ is independent of x, i.e., $\Phi(t,x,u)=\Phi(t,u)$, the first Itô formula is from [QW14, Theorem 3.10].

Proposition A.1.4. *Let BSPDE (A.1) hold in the weak sense with $u|_{\partial\mathcal{O}}=0$. Then there holds almost surely that*
$$\begin{aligned} &\int_{\mathcal{O}}\Phi(t,u(t,x))\,dx+\frac{1}{2}\int_t^T\langle\Phi''(s,u(s)),|v(s)|^2\rangle\,ds \\ =&\int_{\mathcal{O}}\Phi(T,G(x))\,dx-\int_t^T\int_{\mathcal{O}}\partial_s\Phi(s,u(s,x))\,dxds+\int_t^T\langle\Phi'(s,u(s)),\bar{f}(s)\rangle\,ds \\ &-\int_t^T\langle\Phi''(s,u(s))\partial_j u(s),a^{ij}(s)\partial_i u(s)+\sigma^{jr}(s)v^r(s)+\bar{g}^j(s)\rangle\,ds \\ &+\int_{[t,T]\times\mathcal{O}}\Phi'(s,u(s,x))\,\mu(ds,dx)-\int_t^T\langle\Phi'(s,u(s)),v^r(s)\rangle\,dW_s^r. \end{aligned}$$

The following Itô formula extends the preceding one to the positive parts of the weak solutions to BSPDEs.

Theorem A.1.5. *Let BSPDE (A.1) hold in the weak sense but with $u^+|_{\partial\mathcal{O}}=0$. Then there holds almost surely that*
$$\begin{aligned} &\int_{\mathcal{O}}\Phi(t,x,u^+(t,x))\,dx+\frac{1}{2}\int_t^T\langle\Phi''(s,u^+(s)),|v^u(s)|^2\rangle\,ds \\ =&\int_{\mathcal{O}}\Phi(T,x,G^+(x))\,dx-\int_t^T\int_{\mathcal{O}}\partial_s\Phi(s,x,u^+(s,x))\,dxds \\ &+\int_t^T\langle\Phi'(s,u^+(s)),\bar{f}^u(s)\rangle\,ds+\int_t^T\int_{\mathcal{O}}\Phi'(s,x,u^+(s,x))\,\mu(dsdx) \\ &-\int_t^T\langle\Phi''(s,u^+(s))\partial_j u^+(s)+\partial_j\Phi'(s,u^+(s)),a^{ij}(s)\partial_i u^+(s)+\sigma^{j,r}(s)v^{r,u}(s)+\bar{g}^{j,u}(s)\rangle\,ds \\ &-\int_t^T\langle\Phi'(s,u^+(s)),v^{r,u}(s)\rangle\,dW_s^r, \end{aligned} \tag{A.2}$$

where

$$v^{r,u} = 1_{\{u>0\}}v^r, \quad \bar{f}^u = 1_{\{u>0\}}\bar{f}, \quad \bar{g}^{j,u} = 1_{\{u>0\}}\bar{g}^j.$$

Proof. Note that in general we cannot get $u|_{\partial\mathcal{O}} = 0$ from $u^+|_{\partial\mathcal{O}} = 0$, so Proposition A.1.4 is not applicable here. Here, we shall apply an approximation scheme similar to that for [QW14, Theorem 3.10]

Let $\check{u}$ be the stochastic regular parabolic potential (see next subsection for the definition) associated with μ. Now define

$$\begin{cases} -d\hat{u}(t,x) = \big(-\Delta\hat{u}(t,x) + \bar{f}(t,x) + \nabla\cdot\hat{g}(t,x)\big)\,dt - v^r(t,x)dW_t^r, \\ \qquad\qquad (t,x)\in Q, \\ \quad \hat{u}(0,x) = u(0,x), \quad x\in\mathcal{O}, \end{cases}$$

where $\hat{g}^j(t,x) = \partial_j u(t,x) + a^{ij}\partial_i u(t,x) + \sigma^{jr}v^r(t,x) + \bar{g}^j(t,x)$. Then, $u = \hat{u} - \check{u}$ and the zero Dirichlet conditions of u^+ and $\check{u}$ imply $\hat{u}^+|_{\partial\mathcal{O}} = 0$. By [QW14, Proposition 3.9(i)] u is almost surely quasi-continuous. So the integral terms w.r.t. μ in (A.2) is well defined. We can also check that all the other terms in (A.2) are well defined.

Thus by Proposition 3.9(iv) and Remark 3.7 in [QW14], there exist $f^n \in \mathcal{L}^2([0,T];(H^{-1})^+(\mathcal{O}))$, $\check{v}^n \in \mathcal{L}^2([0,T];(L^2(\mathcal{O}))^m)$, $\check{u}^n \in \mathcal{U}(-\infty, f_1^n, g_1^n, G_1^n)$ and $\phi^n \in \mathcal{U}(-\infty, f_2^n, g_2^n, G_2^n)$, for some $f_i^n \in \mathcal{L}^2([0,T];L^2(\mathcal{O}))$, $g_i^n \in \mathcal{L}^2([0,T];(L^2(\mathcal{O}))^n)$, $G_i^n \in \mathcal{L}^2(\mathcal{O})$, $i=1,2$, such that $\phi^n \downarrow 0$ as $n\to\infty$, $dt\times dx\times d\mathbb{P}$ a.e., $\lim_{n\to\infty}\sum_{i=1}^m E\int_0^T \|\check{v}^{n,i}(t)\|^2\,dt = 0$, $\lim_{n\to\infty}\|\check{u}^n - \check{u}\|_{\mathcal{L}^2(\mathcal{K})} = 0$, $\lim_{n\to\infty}(\|f_2^n + \nabla\cdot g_2^n\|_{\mathcal{L}^2([0,T];H^{-1}(\mathcal{O}))} + \|G_2^n\|_{\mathcal{L}^2(\mathcal{O})}) = 0$, $|\check{u}^n - \check{u}| \le \phi^n$ $dt\times dx\times d\mathbb{P}$ a.e., with $\check{u}^n$ satisfying the SPDE

$$\begin{cases} d\check{u}^n(t,x) = & [\Delta\check{u}^n(t,x) + f^n(t,x)]\,dt + \check{v}^n(t,x)\,dW_t, \quad (t,x)\in Q \\ \check{u}^n(0,x) = & 0, \quad x\in\mathcal{O}, \\ \check{u}^n|_{\partial\mathcal{O}} = & 0. \end{cases}$$

Define $u^n := \hat{u} - \check{u}^n$. Then

$$du^n(t,x) = -(-\Delta u^n(t,x) + \bar{f}(t,x) + f^n(t,x) + \nabla\cdot\hat{g}(t,x))\,dt + (v^r(t,x) - \check{v}^{n,r}(t,x))\,dW_t^r.$$

Moreover, $|(u^n)^+ - u^+| \le \phi^n$ $dt\times dx\times d\mathbb{P}$ a.e.. The zero Dirichlet conditions of $\check{u}^n$ and $\hat{u}^+$ imply $(u^n)^+|_{\partial\mathcal{O}} = 0$. By [QT12, Lemma 3.5], we have almost surely

$$\begin{aligned} &\int_{\mathcal{O}} \Phi(t,x,(u^n)^+(t,x))\,dx + \frac{1}{2}\int_t^T \langle \Phi''(s,(u^n)^+(s)), |(v(s) - \check{v}^n(s))1_{\{u^n>0\}}|^2\rangle\,ds \\ =& \int_{\mathcal{O}} \Phi(T,x,(u^n)^+(T,x))\,dx - \int_t^T\int_{\mathcal{O}} \partial_s\Phi(s,x,(u^n)^+(s,x))\,dxds \\ &+ \int_t^T \langle \Phi'(s,(u^n)^+(s)), \bar{f}(s)1_{\{u^n>0\}}\rangle\,ds + \int_t^T \langle \Phi'(s,(u^n)^+(s)), f^n(s)1_{\{u^n>0\}}\rangle_{1,-1}\,ds \\ &- \int_t^T \langle \Phi''(s,(u^n)^+(s))\partial_j(u^n)^+(s) + \partial_j\Phi'(s,(u^n)^+(s)), -\partial_j(u^n)^+(s) + \hat{g}^j(s)1_{\{u^n>0\}}\rangle\,ds \\ &- \int_t^T \langle \Phi'(s,(u^n)^+(s)), (v^r(s) - \check{v}^{n,r}(s))1_{\{u^n>0\}}\rangle\,dW_s^r, \quad \forall t\in[0,T]. \end{aligned} \tag{A.3}$$

By [QW14, Corollary 3.5], there exists $\bar{u} \in \mathcal{L}^2(\mathcal{P})$ such that

$$|\hat{u}| + \phi^1 \leq \bar{u}, \quad dt \times dx \times d\mathbb{P} \ a.e.. \tag{A.4}$$

By (A.4) and the properties (2) and (4) of Φ, there holds $dt \times dx \times d\mathbb{P}$ a.e. that

$$\begin{aligned} |\Phi'(t,x,(u^n)^+(t,x))| &= |\Phi'(t,x,(u^n)^+(t,x)) - \Phi'(t,x,0)| \\ &\leq C|u^n(t,x)| \\ &= C|\hat{u}(t,x) - \check{u}^n(t,x)| \\ &\leq C|\hat{u}(t,x)| + C|\check{u}(t,x)| + C|\check{u}(t,x) - \check{u}^n(t,x)| \\ &\leq C|\hat{u}(t,x)| + C|\check{u}(t,x)| + C\phi^n(t,x) \\ &\leq C(|\check{u}(t,x)| + \bar{u}(t,x)). \end{aligned} \tag{A.5}$$

By property (4) of Φ, there holds $dt \times dx \times d\mathbb{P}$ a.e. that

$$\begin{aligned} |\Phi'(t,x,(u^n)^+(t,x)) - \Phi'(t,x,u^+(t,x))| &\leq C|(u^n)^+(t,x) - u^+(t,x)| \\ &\leq C|\check{u}^n(t,x) - \check{u}(t,x)| \\ &\leq C\phi^n(t,x). \end{aligned} \tag{A.6}$$

(A.5), (A.6) and [QW14, Proposition 3.9(ii)] yield that

$$\begin{aligned} &\lim_{n\to\infty} \int_t^T \langle \Phi'(s,(u^n)^+(s)), f^n(s) 1_{\{u^n>0\}}\rangle_{1,-1}\, ds \\ = &\int_{\mathcal{O}_t} \Phi'(s,x,u^+(s,x))\, \mu(dsdx) \ a.s.. \end{aligned}$$

Moreover,

$$\begin{aligned} &\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_t^T \langle \Phi'(s,(u^n)^+(s)), (v^r(s) - \check{v}^{n,r}(s)) 1_{\{u^n>0\}}\rangle\, dW_s^r \right.\right. \\ &\quad \left.\left. - \int_t^T \langle \Phi'(s,u^+(s)), v^r(s) 1_{\{u>0\}}\rangle\, dW_s^r\right|\right] \\ &\leq C\mathbb{E}\left(\int_0^T \left|\langle \Phi'(s,(u^n)^+(s)), (v^r(s) - \check{v}^{n,r}(s)) 1_{\{u^n>0\}}\rangle \right.\right. \\ &\quad \left.\left. - \langle \Phi'(s,u^+(s)), v^r(s) 1_{\{u>0\}}\rangle\right|^2 ds\right)^{\frac{1}{2}} \\ &\leq C\mathbb{E}\left(\int_0^T \left|\langle \Phi'(s,(u^n)^+(s)), v(s) 1_{\{u^n>0\}}\rangle - \langle \Phi'(s,u^+(s)), v(s) 1_{\{u>0\}}\rangle\right|^2 ds\right)^{\frac{1}{2}} \\ &\quad + C\mathbb{E}\left(\int_0^T \left|\langle \Phi'(s,(u^n)^+(s)) - \Phi'(s,0), \check{v}^n(s) 1_{\{u^n>0\}}\rangle\right|^2 ds\right)^{\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
\leq & C\left(\mathbb{E}\mathrm{esssup}_{0\leq t\leq T}\|(u^n)^+(t)-u^+(t)\|^2\right)^{\frac{1}{2}}\left(\mathbb{E}\int_0^T\|v(t)\|^2\,dt\right)^{\frac{1}{2}}\\
& + C\left(\mathbb{E}\mathrm{esssup}_{0\leq t\leq T}\|(u^n)^+(t)\|^2\right)^{\frac{1}{2}}\left(\mathbb{E}\int_0^T\|v(t)(1_{\{u^n>0\}}-1_{\{u>0\}})\|^2\,dt\right)^{\frac{1}{2}}\\
& + C\left(\mathbb{E}\mathrm{esssup}_{0\leq t\leq T}\|u^n(t)\|^2\right)^{\frac{1}{2}}\left(\mathbb{E}\int_0^T\|\check{v}^n(t)\|^2\,dt\right)^{\frac{1}{2}}\\
\leq & C\left(\mathbb{E}\|u^n-u\|_{\mathcal{K}}^2\right)^{\frac{1}{2}}\left(\mathbb{E}\int_0^T\|v(t)\|^2\,dt\right)^{\frac{1}{2}}\\
& + C\left(\mathbb{E}\|u^n\|_{\mathcal{K}}^2\right)^{\frac{1}{2}}\left(\mathbb{E}\int_0^T\|v(t)(1_{\{u^n>0\}}-1_{\{u>0\}})\|^2\,dt\right)^{\frac{1}{2}}\\
& + C\left(\mathbb{E}\mathrm{esssup}_{0\leq t\leq T}\|u^n(t)\|^2\right)^{\frac{1}{2}}\left(\mathbb{E}\int_0^T\|\check{v}^n(t)\|^2\,dt\right)^{\frac{1}{2}}\\
\to & \, 0.
\end{aligned}
$$

By the properties of Φ and the fact that $|(u^n)^+-u^+|\leq\phi^n\ dt\times dx\times d\mathbb{P}$ a.e., the convergence of other terms can be treated analogously. Finally by letting $n\to\infty$, we obtain almost surely that

$$
\begin{aligned}
& \int_{\mathcal{O}}\Phi(t,x,u^+(t,x))\,dx+\frac{1}{2}\int_t^T\langle\Phi''(s,u^+(s)),|(v(s)1_{\{u>0\}}|^2\rangle\,ds\\
=& \int_{\mathcal{O}}\Phi(T,x,u^+(T,x))\,dx-\int_t^T\int_{\mathcal{O}}\partial_s\Phi(s,x,u^+(s,x))\,dxds\\
& +\int_t^T\langle\Phi'(s,u^+(s)),\bar{f}(s)1_{\{u>0\}}\rangle\,ds+\int_t^T\int_{\mathcal{O}}\Phi'(s,x,u^+(s,x))\,\mu(dsdx)\\
& -\int_t^T\langle\Phi''(s,u^+(s))\partial_j u^+(s)+\partial_j\Phi'(s,u^+(s)),-\partial_j u^+(s)+\hat{g}^j(s)1_{\{u^n>0\}}\rangle\,ds\\
& -\int_t^T\langle\Phi'(s,u^+(s)),v^r(s)1_{\{u>0\}}\rangle\,dW_s^r,\quad\forall t\in[0,T].
\end{aligned}
$$

□

A.2. Some definitions associated with stochastic regular measures

In general the random measure μ in (2.1) can be a local time, which is not absolutely continuous w.r.t. Lebesgue measure. Hence, the Skorokhod condition $\int_Q(u-\xi)\,\mu(dt,dx)=0$ might not make sense. To give a precise meaning to the Skorohod condition, the theory of parabolic potential and capacity introduced by

[Pie79, Pie80] was generalized by [QW14] to a backward stochastic framework. This subsection recalls the notion of quasi continuity and stochastic regular measure, which are repeatedly used in the main text and in the proof of Theorem A.1.5. Moreover, spaces used in the proof of Theorem A.1.5 are also presented.

First some spaces are introduced. Denote by $H_0^1(\mathcal{O})$ the first order Sobolev space vanishing on the boundary $\partial\mathcal{O}$ equipped with the norm $\|v\|_1^2 := \|v\|^2 + \|\nabla v\|^2$ and by $H^{-1}(\mathcal{O})$ the dual space of $H_0^1(\mathcal{O})$. The dual pair between $H_0^1(\mathcal{O})$ and $H^{-1}(\mathcal{O})$ is denoted by $\langle\cdot,\cdot\rangle_{1,-1}$. Define $(H^{-1})^+(\mathcal{O}) = \{v \in H^{-1}(\mathcal{O}) : \langle\varphi, v\rangle_{1,-1} \geq 0, \text{ for each } \varphi \in H_0^1(\mathcal{O}) \text{ and } \varphi \geq 0\}$.

For a Hilbert space V, denote by $\mathcal{L}^2([0,T];V)$ the set of all $L^2([0,T];V)$ valued $(\mathcal{F}_t)$ adapted process u with the norm defined as $\|u\|_{\mathcal{L}^2([0,T];V)} := \left(E\|u\|^2_{L^2([0,T];V)}\right)^{\frac{1}{2}} < \infty$. Denote by $\mathcal{L}^2(\mathcal{O})$ the set of all $L^2(\mathcal{O})$ valued $(\mathcal{F}_t)$ adapted process u with the norm $\|u\|_{\mathcal{L}^2(\mathcal{O})} := \left(E\|u\|^2\right)^{\frac{1}{2}} < \infty$

Denote $\mathcal{K} := L^\infty([0,T];L^2(\mathcal{O})) \cap L^2([0,T];H_0^1(\mathcal{O}))$, equipped with the norm

$$\|v\|_{\mathcal{K}} := \left(\|v\|^2_{L^\infty([0,T];L^2(\mathcal{O}))} + \|v\|^2_{L^2([0,T];H_0^1(\mathcal{O}))}\right)^{\frac{1}{2}}.$$

Set $\mathcal{W} = \{v \in L^2(0,T;H_0^1) : \partial_t v \in L^2(0,T;H^{-1})\}$ endowed with the norm

$$\|v\|_{\mathcal{W}} = \left(\|v\|^2_{L^2(0,T;H_0^1)} + \|\partial_t v\|^2_{L^2(0,T;H^{-1})}\right)^{\frac{1}{2}},$$

where H^{-1} is the dual space of H_0^1. Furthermore, we set

$$\mathcal{W}_T = \{v \in \mathcal{W} : v(T) = 0\},\ \mathcal{W}^+ = \{v \in \mathcal{W} : v \geq 0\},\ \mathcal{W}_T^+ = \mathcal{W}_T \cap \mathcal{W}^+.$$

Definition A.2.1. We denote by $\mathcal{P}$ the set of parabolic potentials, which is the class of $v \in \mathcal{K}$ such that

$$\int_0^T -\langle\partial_t\varphi(t), v(t)\rangle\,dt + \int_0^T \langle\partial_i\varphi(t), \partial_i v(t)\rangle\,dt \geq 0,\ \forall\varphi \in \mathcal{W}_T^+.$$

Denote by $\mathcal{C}(Q)$ the class of continuously differentiable functions in Q with compact support. By the Hahn-Banach theorem and because $\mathcal{C}(Q) \cap \mathcal{W}_T$ is dense in $\mathcal{C}(Q)$, parabolic potentials can be represented by associated Radon measures. This leads to the following proposition, due to Pierre [Pie80].

Proposition A.2.2. *Let $v \in \mathcal{P}$. Then there exists a unique Radon measure on $[0,T) \times \mathcal{O}$, denoted by μ^v, such that*

$$\forall\ \varphi \in \mathcal{W}_T \cap \mathcal{C}(Q),\ \int_0^T -\langle\partial_t\varphi(t), v(t)\rangle + \int_0^T \langle\partial_i\varphi(t), \partial_i v(t)\rangle\,dt = \int_0^T \int_{\mathcal{O}} \varphi(t,x)\mu^v(dt,dx)$$

Definition A.2.3. For any open set $A \subset [0,T) \times \mathcal{O}$, the parabolic capacity of A is defined as

$$\text{cap}(A) = \inf\{\|\varphi\|^2_{\mathcal{W}} : \varphi \in \mathcal{W}^+,\ \varphi \geq 1\ a.e. \text{ on } A\}.$$

For any Borel set $B \subset [0,T) \times \mathcal{O}$, its parabolic capacity is defined as

$$\operatorname{cap}(B) = \inf\{\operatorname{cap}(A) : A \supset B,\ A \text{ is open}\}.$$

Definition A.2.4. A real valued function ϕ on $[0,T) \times \mathcal{O}$ is said to be quasi-continuous, if there exists a sequence of non-increasing open sets $A_n \subset [0,T) \times \mathcal{O}$ such that

(1) ϕ is continuous on the complement of each A_n;

(2) $\lim_{n\to\infty} \operatorname{cap}(A_n) = 0$.

Denote by $\mathcal{P}_0$ the class of $v \in \mathcal{P}$ such that v is quasi-continuous and $v(0) = 0$ in L^2. Each element $v \in \mathcal{P}_0$ is called a regular potential and the associated Radon measure in Definition A.2.2 is called a regular measure. Furthermore, let $\mathcal{L}^0(\mathcal{K})$ be the class of the measurable maps from $(\Omega, \mathcal{F}_T)$ to $\mathcal{K}$, such that each element $v \in \mathcal{L}^0(\mathcal{K})$ is an L^2 valued adapted process. $\mathcal{L}^0(\mathcal{P})$ and $\mathcal{L}^0(\mathcal{P}_0)$ are similarly defined as $\mathcal{L}^0(\mathcal{K})$. Moreover, set

$$\mathcal{L}^2(\mathcal{K}) := L^2(\Omega, \mathcal{F}_T; \mathcal{K}) \cap \mathcal{L}^0(\mathcal{K})$$

endowed with the norm

$$\|v\|_{\mathcal{L}^2(\mathcal{K})} = \left(E\|v\|^2_{\mathcal{K}}\right)^{1/2}.$$

The stochastic parabolic potential is defined as

$$\mathcal{L}^2(\mathcal{P}) := \mathcal{L}^2(\mathcal{K}) \cap \mathcal{L}^0(\mathcal{P}),$$

endowed with the norm

$$\|u\|_{\mathcal{L}^2(\mathcal{P})} = \|u\|_{\mathcal{L}^2(\mathcal{K})}.$$

In addition, we define the stochastic regular parabolic potential as

$$\mathcal{L}^2(\mathcal{P}_0) := \mathcal{L}^2(\mathcal{P}) \cap \mathcal{L}^0(\mathcal{P}_0),$$

and the associated random Radon measure is called a stochastic regular measure.

A.3. Wasserstein distance and representation of martingales

Definition A.3.1. Let (E, ϱ) be a metric space. Denote by $\mathcal{P}_p(E)$ the class of all probability measures on E with finite moment of p-th order. The *p-th Wasserstein metric* on $\mathcal{P}_p(E)$ is defined by:

$$\begin{aligned}&\mathcal{W}_{p,(E,\varrho)}(\mathbb{P}_1, \mathbb{P}_2)\\ &= \inf\left\{\left(\int_{E\times E} \varrho(x,y)^p \, \gamma(dx,dy)\right)^{\frac{1}{p}} : \gamma(dx, E) = \mathbb{P}_1(dx), \gamma(E, dy) = \mathbb{P}_2(dy)\right\}.\end{aligned} \tag{A.7}$$

The set $\mathcal{P}_p(E)$ endowed with the Wasserstein distance is denoted by $\mathcal{W}_{p,(E,\varrho)}$ or $\mathcal{W}_{p,E}$ or $\mathcal{W}_p$ if there is no risk of confusion about the underlying state space or distance.

It is well known [EKM90, Theorem III-10] that for every continuous square integrable martingale m with quadratic variation process $\int_0^\cdot \int_U a(s,u)\, v_s(du)ds$, where $a = \sigma\sigma^\top$ and σ is a bounded measurable function and v is $\mathcal{P}(U)$ valued stochastic process, on some extension of the original probability space, there exists a martingale measure M with intensity $v_s(du)ds$ such that $m_\cdot = \int_0^\cdot \int_U \sigma(t,u)\, M(du,dt)$. This directly leads to the following proposition, which is frequently used in the main text.

Proposition A.3.2. *The existence of solution $\mathbb{P}$ to the martingale problem (3.6) is equivalent to the existence of the weak solution to the following SDE*

$$d\bar{X}_t = \int_U b(t, \bar{X}_t, \mu_t, u)\, \bar{Q}_s(du)ds + \int_U \sigma(t, \bar{X}_t, \mu_t, u)\, \bar{M}(du, dt) + c(t)\, d\bar{Z}_t, \tag{A.8}$$

where $\bar{X}$, $\bar{M}$ and $\bar{Z}$ are defined on some extension $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ and $\bar{M}$ is a martingale measure with intensity $\bar{Q}$. Moreover, the two solutions are related by $\mathbb{P} = \bar{\mathbb{P}} \circ (\bar{X}, \bar{Q}, \bar{Z})^{-1}$.

A.4. Strong M_1 Topology in Skorokhod Space

In this section, we summarise some definitions and properties about strong Skorokhod M_1 topology. For more details, please refer to Chapter 3, 11 and 12 in [Whi02]. Note that in [Whi02] two M_1 topologies are introduced, the strong one and the weak one. In this thesis, we only apply the strong one. So without abuse of terminologies, we just take M_1 topology for short.

For $x \in \mathcal{D}(0,T)$, denote by $Disc(x)$ the set of discontinuous points of x. Note that on $[0,T]$, $Disc(x)$ is at most countable. Define the thin graph of x as

$$G_x = \{(z,t) \in \mathbb{R}^d \times [0,T] : z \in [x_{t-}, x_t]\}, \tag{A.9}$$

where x_{t-} is the left limit of x at t and $[a,b]$ means the line segment between a and b, i.e., $[a,b] = \{\alpha a + (1-\alpha)b : 0 \le \alpha \le 1\}$. On the thin graph, we define an order relation. For each pair $(z_i, t_i) \in G_x$, $i = 1,2$, $(z_1,t_1) \le (z_2,t_2)$ if either of the following holds: (1) $t_1 < t_2$; (2) $t_1 = t_2$ and $|z_1 - x_{t_1-}| < |z_2 - x_{t_2-}|$.

Now we define the parameter representation, on which the M_1 topology depends. The mapping pair (u,r) is called a parameter representation if $(u,r) : [0,1] \to G_x$, which is continuous and nondecreasing w.r.t. the order relation defined above. Denote by Π_x all the parameter representations of x. Let

$$d_{M_1}(x_1,x_2) = \inf_{(u_i,r_i)\in\Pi_{x_i}, i=1,2} ||u_1 - u_2|| \vee ||r_1 - r_2||. \tag{A.10}$$

It can be shown that d_{M_1} is a metric on $\mathcal{D}(0,T)$ such that $\mathcal{D}(0,T)$ is a Polish space. The topology induced by d_{M_1} is called M_1 topology.

For each $t \in [0,T]$ and $\delta > 0$, the oscillation function around t is defined as

$$\bar{v}(x,t,\delta) = \sup_{0\vee(t-\delta)\le t_1 \le t_2 \le (t+\delta)\wedge T} |x_{t_1} - x_{t_2}|, \tag{A.11}$$

and the so called strong M_1 oscillation function is defined as

$$w_s(x,t,\delta) = \sup_{0\vee(t-\delta)\le t_1 < t_2 < t_3 \le (t+\delta)\wedge T} |x_{t_2} - [x_{t_1}, x_{t_3}]|, \tag{A.12}$$

where $|x_{t_2} - [x_{t_1}, x_{t_3}]|$ is the distance from x_{t_2} to the line segment $[x_{t_1}, x_{t_3}]$. Moreover,

$$w_s(x,\delta) := \sup_{0\le t\le T} w_s(x,t,\delta). \tag{A.13}$$

Proposition A.4.1. *The following statements about the characterization of M_1 convergence are equivalent,*

1. $x^n \to x$ *in M_1 topology;*

2. *there exist $(u,r) \in \Pi_x$ and $(u^n, r^n) \in \Pi_{x^n}$ for each n such that*

$$\lim_{n\to\infty} \|u^n - u\| \vee \|r^n - r\| = 0;$$

3. $x_n(t) \to x(t)$ *for each* $t \in [0,T] \setminus Disc(x)$ *including* 0 *and* T*, and*

$$\lim_{\delta\to 0} \overline{\lim}_{n\to\infty} w_s(x^n,\delta) = 0.$$

Moreover, each one of the above three items implies the local uniform convergence of x^n *to* x *at each continuous point of* x*, that is, for each* $t \notin Disc(x)$*, there holds*

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \sup_{t-\delta\le s\le t+\delta} |x_n(s) - x(s)| = 0. \tag{A.14}$$

Remark A.4.2. Proposition A.4.1, 3. implies that $(\mathcal{D}(0,T), d_{M_1})$ convergence is stronger than $L^\alpha[0,T]$ convergence, for any $\alpha > 0$. In fact, if $x^n \to x$ in M_1, then $x^n_t \to x_t$ for a.e. $t \in [0,T]$, due to Proposition A.4.1, 3. Moreover,

$$|x^n_t - x_t|^\alpha \le 2^\alpha \left(d^\alpha_{M_1}(x^n,0) + d^\alpha_{M_1}(x,0)\right) \to 2^{\alpha+1} d_{M_1}(x,0) < \infty.$$

Thus, the assertion follows from dominated convergence.

Proposition A.4.3. *A subset* A *of* $(\mathcal{D}(0,T), d_{M_1})$ *is relatively compact w.r.t.* M_1 *topology if and only if*

$$\sup_{x\in A} ||x|| < \infty \tag{A.15}$$

and

$$\lim_{\delta\downarrow 0} \sup_{x\in A} w'_s(x,\delta) = 0, \tag{A.16}$$

where

$$w'_s(x,\delta) = w_s(x,\delta) \vee \bar{v}(x,0,\delta) \vee \bar{v}(x,T,\delta). \tag{A.17}$$

In [Whi02], it is assumed that $x_{0-} = x_0$, which implies there is no jump at the initial time. For singular control problems it is natural to admit jumps a the initial time. It is also implied by Proposition A.4.3 that the terminal time T is a continuous point of $x \in \mathcal{D}(0,T)$. This, too, is not appropriate for singular control problems. In order to adapt the relative compactness criteria stated in Proposition A.4.3 to functions with jumps at 0 and T, we work on the extended state spaces $\widetilde{\mathcal{D}}(\mathbb{R})$ and $\widetilde{\mathcal{A}}(\mathbb{R})$. Convergence in $\widetilde{\mathcal{D}}(\mathbb{R})$ can be defined as convergence in $\mathcal{D}(\mathbb{R})$, where a sequence $\{x^n, n \ge 1\}$ converges to x in $\mathcal{D}(\mathbb{R})$ if and only if the sequences $\{x^n|_{[a,b]}, n \ge 1\}$ converge to $x|_{[a,b]}$ for all $a < b$ at which x is continuous; see [Whi02, Chapter 3].

Relative compactness of a sequence $\{x^n, n \ge 1\} \subseteq \widetilde{\mathcal{D}}(\mathbb{R})$ is equivalent to that of the sequence $\{x^n|_{[a,b]}, n \ge 1\} \subseteq \mathcal{D}[a,b]$ for any $a < 0$ and $b > T$. Specifically, we have the following result.

Proposition A.4.4. *The sequence $\{x^n, n \geq 1\} \subseteq \widetilde{\mathcal{D}}(\mathbb{R})$ is relatively compact if and only if*

$$\sup_n ||x_n|| < \infty \quad \text{and} \quad \lim_{\delta \downarrow 0} \sup_{x \in A} \widetilde{w}_s(x, \delta) = 0, \tag{A.18}$$

where the modified oscillation function $\widetilde{w}_s$ is defined as

$$\widetilde{w}_s(x, \delta) = w_s(x, \delta) + \sup_{0 \leq s < t \leq \delta} |x_s - [0, x_t]|. \tag{A.19}$$

Corollary A.4.5. *Let $A = \{z \in \widetilde{\mathcal{A}}(\mathbb{R}) : z_T \leq K\}$ for some $K > 0$. Then A is $(\widetilde{\mathcal{D}}(\mathbb{R}), M_1)$ compact.*

Proof. This follows from Proposition A.4.4 as $w_s(z, t, \delta) = 0$ for each $z \in A$, $t \in \mathbb{R}$ and $\delta > 0$. $\square$

We notice that the modified oscillation function $\widetilde{w}_s$ is defined in terms of the original oscillation function w_s and the line segment (if it exists) between $0-$ and 0[1]. As such the space $\widetilde{\mathcal{D}}(\mathbb{R})$ is isomorphic to the space

$$\mathcal{D}_{0,T} := \{(y, x|_{[0,T]}) \in \mathbb{R}^d \times \mathcal{D}(0, T) : x \in \mathcal{D}(\mathbb{R}), x_{0-} = y\}.$$

On $\mathcal{D}_{0,T}$, we can construct the modified thin graph by taking the segment (if it exists) between $0-$ and 0 into consideration. In the same spirit of M_1 metric on the thin graph, we can define the modified M_1 metric (we call it $\widetilde{M_1}$) on the modified thin graph. Therefore, we have the following characterization of convergence in $(\mathcal{D}_{0,T}, \widetilde{M_1})$.

Lemma A.4.6. *$(y^n, x^n|_{[0,T]}) \to (y, x|_{[0,T]})$ in $\widetilde{M_1}$ on $\mathcal{D}_{0,T}$ if and only if $x^n_t \to x_t$ for each $t \in [0, T] \setminus Disc(x)$ including T, $y^n \to y$, and*

$$\lim_{\delta \to 0} \overline{\lim}_{n \to \infty} \widetilde{w}_s(x^n, \delta) = 0.$$

For each set $A \subseteq \mathcal{D}_{0,T}$, define

$$\begin{aligned}\widetilde{A} := \{&\widetilde{x} \in \widetilde{\mathcal{D}}(\mathbb{R}) : \text{for some } (y, x|_{[0,T]}) \in A,\ \widetilde{x}|_{[0,T]} = x|_{[0,T]},\\ &\widetilde{x}_t = y \text{ if } t < 0 \text{ and } \widetilde{x}_t = x_T \text{ if } t > T\}.\end{aligned}$$

It is easy to show that the $(\mathcal{D}_{0,T}, \widetilde{M_1})$ relative compactness of A is equivalent to the $(\widetilde{\mathcal{D}}(\mathbb{R}), M_1)$ relative compactness of $\widetilde{A}$. In this way, we could consider $\mathcal{D}_{0,T}$ as the canonical space as well.

Proposition A.4.7. *A sequence of probability measures $\{\mathbb{P}_n\}_{n \geq 1}$ on $\widetilde{\mathcal{D}}(\mathbb{R})$ is tight if and only if*
(1) for each $\epsilon > 0$, there exists c large enough such that

$$\sup_n \mathbb{P}_n(||x|| > c) < \epsilon; \tag{A.20}$$

[1] Due to the right-continuity of the elements in $\widetilde{\mathcal{D}}(\mathbb{R})$ there is no line segment between T and $T+$.

(2) for each $\epsilon > 0$ and $\eta > 0$, there exists $\delta > 0$ small enough such that

$$\sup_n \mathbb{P}_n(\widetilde{w}_s(x,\delta) \geq \eta) < \epsilon. \tag{A.21}$$

The following proposition shows that if two M_1 limits do not jump at the same time, then the M_1 convergence preserves by the addition operation.

Proposition A.4.8. *If $x^n \to x$ and $y^n \to y$ in $(\mathcal{D}(0,T), d_{M_1})$, and $Disc(x) \cap Disc(y) = \emptyset$, then*

$$x^n + y^n \to x + y \text{ in } M_1. \tag{A.22}$$

A.5. Sketch Proof of Proposition 3.2.5

It is sufficient to establish the equivalence of martingale problems in Definition 3.1.5 and Proposition 3.2.5. Only the one-dimensional case is proved; the multi-dimensional case is similar.

Proposition 3.2.5 $\Rightarrow$ Definition 3.1.5: Without loss of generality (see [KS91, Proposition 4.11 and Remark 4.12]), we can take $\phi(y) = y,\ y^2$ and following the proof of [KS91, Proposition 4.6], we have that M is a continuous martingale with the quadratic variation

$$\langle M\rangle_t = \int_0^t \int_U a(s, X_s, \mu_s, u)\, Q_s(du)ds,$$

where

$$M_t = Y_t - \int_0^t \int_U b(s, X_s, \mu_s, u)\, Q_s(du)ds.$$

By applying Itô's formula to $\phi(X_t)$ and noting $X = Y + \int_0 c(s)\, dZ_s$, the desired result follows from

$$\begin{aligned}\phi(X_t) =& \phi(X_{0-}) + \int_0^t \phi'(X_s)\, dM_s + \int_0^t \int_U \phi'(X_s) b(s, X_s, \mu_s, u)\, Q_s(du)ds \\ &+ \frac{1}{2}\int_0^t \int_U \phi''(X_s) a(s, X_s, \mu_s, u)\, Q_s(du)ds \\ &+ \int_0^t \phi'(X_{s-}) c(s)\, dZ_s + \sum_{0\le s\le t} \left[\phi(X_s) - \phi(X_{s-}) - \phi'(X_{s-})\triangle X_s\right].\end{aligned}$$

Definition 3.1.5 $\Rightarrow$ Proposition 3.2.5: By Proposition A.3.2, there exists $(\overline{X}, \overline{Q}, \overline{Z})$ and a martingale measure $\overline{M}$ with intensity $\overline{Q}$ on some extension $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$, s.t. (A.8) holds and $\mathbb{P}\circ(X, Q, Z)^{-1} = \overline{\mathbb{P}}\circ(\overline{X}, \overline{Q}, \overline{Z})^{-1}$. Let

$$\overline{Y}_\cdot = \overline{X}_\cdot - \int_0^\cdot c(s)\, d\overline{Z}_s.$$

Then

$$Y_\cdot := X_\cdot - \int_0^\cdot c(s)\, dZ_s \overset{d}{=} \overline{Y}_\cdot.$$

By applying Itô's formula to $\phi(\overline{Y}_t)$,

$$\phi(\overline{Y}_t) - \int_0^t \int_U \phi'(\overline{Y}_s) b(s, \overline{X}_s, \mu_s, u)\, \overline{Q}_s(du)ds - \frac{1}{2}\int_0^t \int_U \phi''(\overline{Y}_s) a(s, \overline{X}_s, \mu_s, u)\, \overline{Q}_s(du)ds$$

is a martingale. Hence the following is also a martingale:

$$\phi(Y_t) - \int_0^t \int_U \phi'(Y_s) b(s, X_s, \mu_s, u)\, Q_s(du)ds - \frac{1}{2}\int_0^t \int_U \phi''(Y_s) a(s, X_s, \mu_s, u)\, Q_s(du)ds.$$

A.6. Estimates for A

Assume that λ, η and $1/\eta$ are bounded.

Lemma A.6.1. *[AJK14, Theorem 2.2][GHS17, Theorem 6.1, Theorem 6.3] In $L^2_{\mathbb{F}}(\Omega; C[0,T^-]) \times L^2_{\mathbb{F}}([0,T^-]; \mathbb{R}^m)$ there exists a unique solution to*

$$\begin{cases} -dA_t = \left(2\lambda_t - \dfrac{A_t^2}{2\eta_t}\right) dt - Z_t^A \, dW_t, \\ \quad A_T = \infty. \end{cases}$$

Moreover, there holds the following estimate

$$\frac{1}{\mathbb{E}\left[\int_t^T \frac{1}{2\eta_s} \, ds \middle| \mathcal{F}_t\right]} \leq A_t \leq \frac{1}{(T-t)^2} \mathbb{E}\left[\int_t^T 2\eta_s + 2(T-s)^2 \lambda_s \, ds \middle| \mathcal{F}_t\right]. \tag{A.23}$$

We also consider A^n the unique bounded solution of the BSDE

$$\begin{cases} -dA_t^n = \left(2\lambda_t - \dfrac{(A_t^n)^2}{2\eta_t}\right) dt - Z_t^{A^n} \, dW_t, \\ \quad A_T^n = 2n. \end{cases}$$

Lemma A.6.2. *The sequence A^n is non decreasing and converges to A. There exists a constant $\mathfrak{C}$ such that for any n:*

$$\|A^n\|_{-1} + \|A^n\|_{n,-1} \leq \mathfrak{C}.$$

Proof. The first assertion is a result of [AJK14, Theorem 2.2]. For any t, n and a, we have

$$2\lambda_t - \frac{a^2}{2\eta_t} \leq 2\lambda_t - \frac{2}{\left(T - t + \frac{\eta_\star}{n}\right)} a + \frac{2\eta_t}{\left(T - t + \frac{\eta_\star}{n}\right)^2} = g(t,a).$$

Let us denote by Ψ^n the solution of the BSDE with generator g and terminal condition $2n$. By the comparison principle for BSDEs, we have $A_t^n \leq \Psi_t^n$ and by the solution formula for linear BSDEs,

$$\begin{aligned} \Psi_t^n &= \left(\frac{T + \frac{\eta_\star}{n}}{T - t + \frac{\eta_\star}{n}}\right)^2 \mathbb{E}\Bigg[\left(\frac{\frac{\eta_\star}{n}}{T + \frac{\eta_\star}{n}}\right)^2 2n \\ &\quad + \int_t^T \left(\frac{T - s + \frac{\eta_\star}{n}}{T + \frac{\eta_\star}{n}}\right)^2 \left(\frac{2\eta_s}{\left(T - s + \frac{\eta_\star}{n}\right)^2} + 2\lambda_s\right) \Bigg| \mathcal{F}_t\Bigg] \\ &= \frac{2\eta_\star^2}{n} \frac{1}{\left(T - t + \frac{\eta_\star}{n}\right)^2} + \frac{1}{\left(T - t + \frac{\eta_\star}{n}\right)^2} \mathbb{E}\left[\int_t^T \left(2\eta_s + 2\left(T - s + \frac{\eta_\star}{n}\right)^2 \lambda_s\right) \middle| \mathcal{F}_t\right]. \end{aligned}$$

Hence

$$
\begin{aligned}
&\left(T-t+\frac{\eta_\star}{n}\right)\Psi_t^n \\
&\le \frac{2\eta_\star^2}{\eta_\star+n(T-t)}+\frac{1}{\left(T-t+\frac{\eta_\star}{n}\right)}\mathbb{E}\left[\int_t^T\left(2\eta_s+2\left(T-s+\frac{\eta_\star}{n}\right)^2\lambda_s\right)\Big|\mathcal{F}_t\right] \\
&\le 2\eta_\star+\frac{1}{T-t}\mathbb{E}\left[\int_t^T\left(2\eta_s+2\left(T-s+\frac{\eta_\star}{n}\right)^2\lambda_s\right)\Big|\mathcal{F}_t\right]=\mathfrak{C}.
\end{aligned}
$$

Thus $\left(T-t+\frac{\eta_\star}{n}\right)A_t^n\le\mathfrak{C}$, that is $\|A^n\|_{n,-1}\le\mathfrak{C}$. □

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Declaration of Independent work

I declare that I have completed the thesis independently using only the aids and tools specified. I have not applied for a doctor's degree in the doctoral subject elsewhere and do not hold a corresponding doctor's degree. I have taken due note of the Faculty of Mathematics and Natural Sciences PhD Regulations, published in the Official Gazette of Humboldt-Universität zu Berlin no. 126/2014 on 18/11/2014.

Berlin, February 6, 2018 Guanxing Fu